Lecture 2

Introduction to Proper Generalized Decomposition Techniques

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UKACM 2021 Conference

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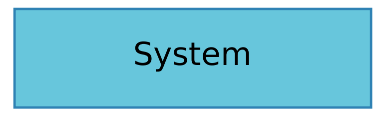
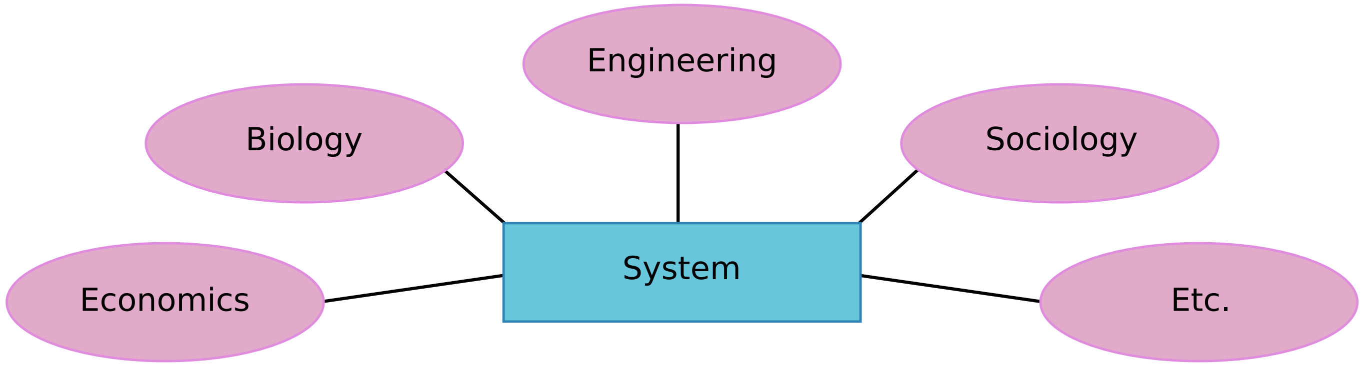
Introduction



Introduction



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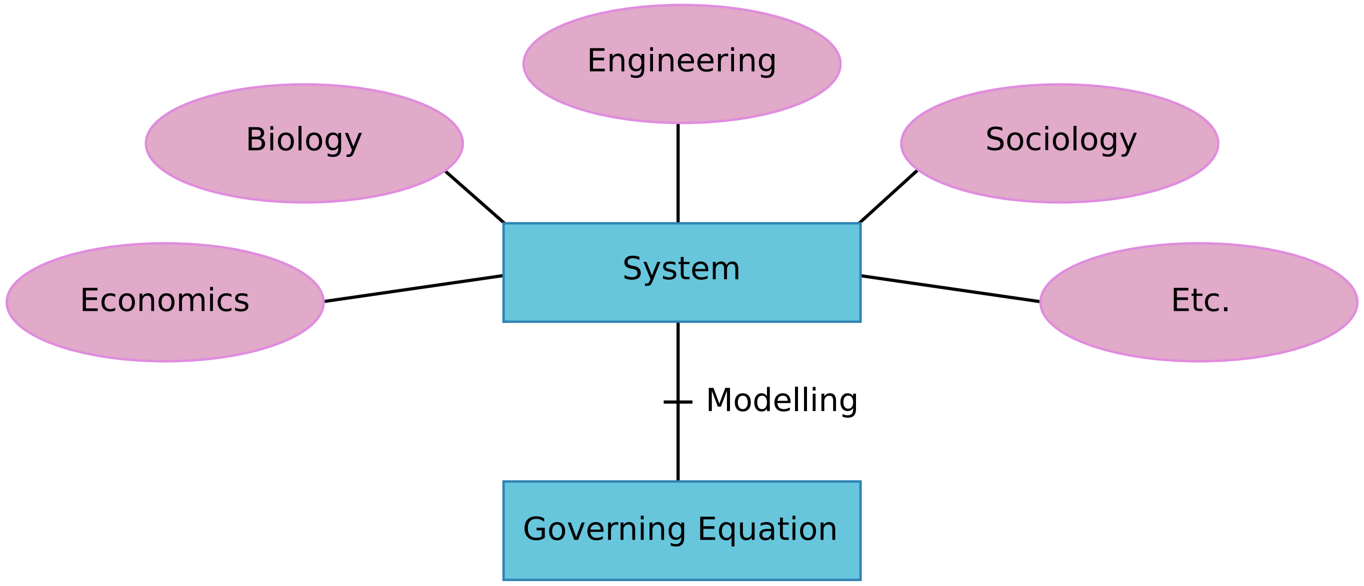

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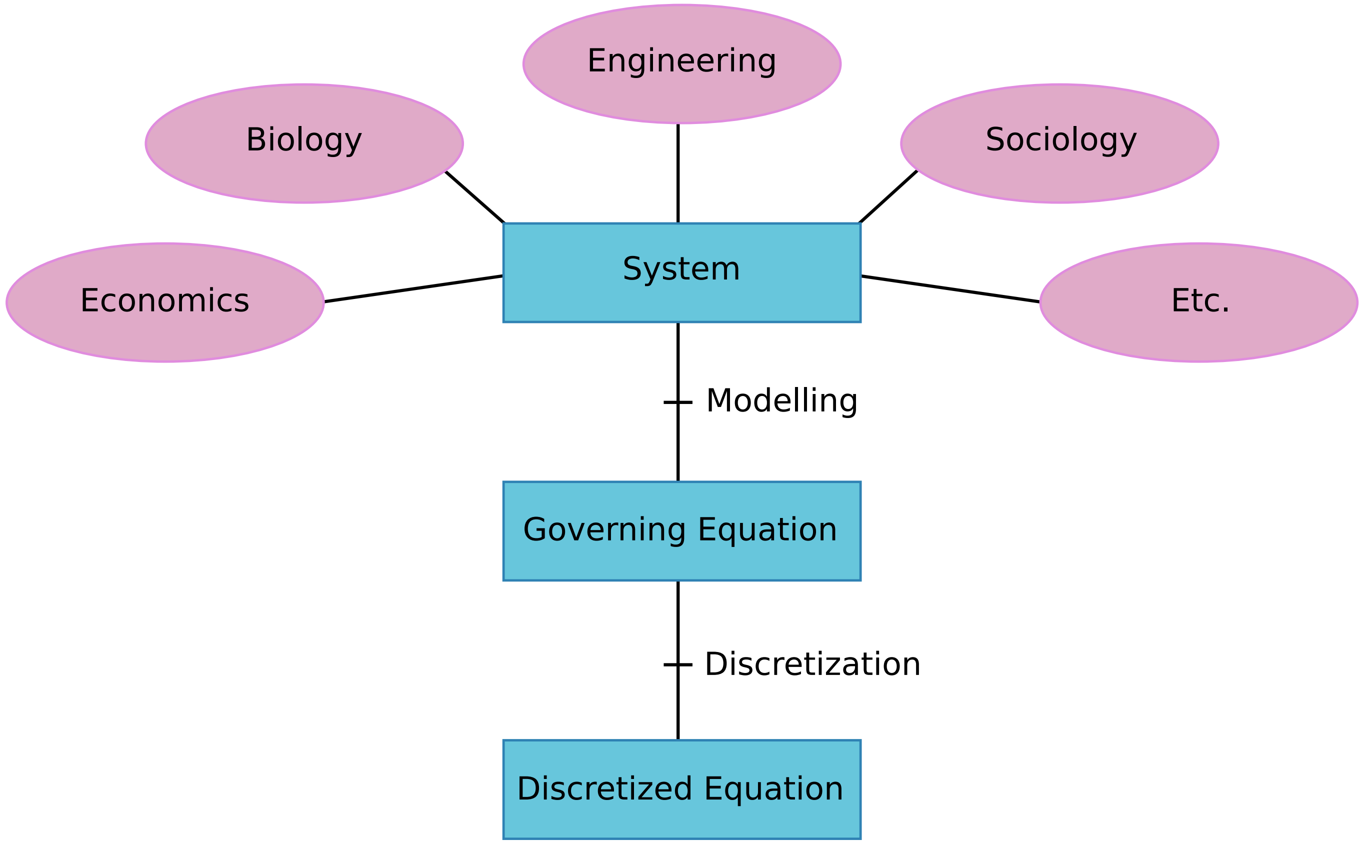

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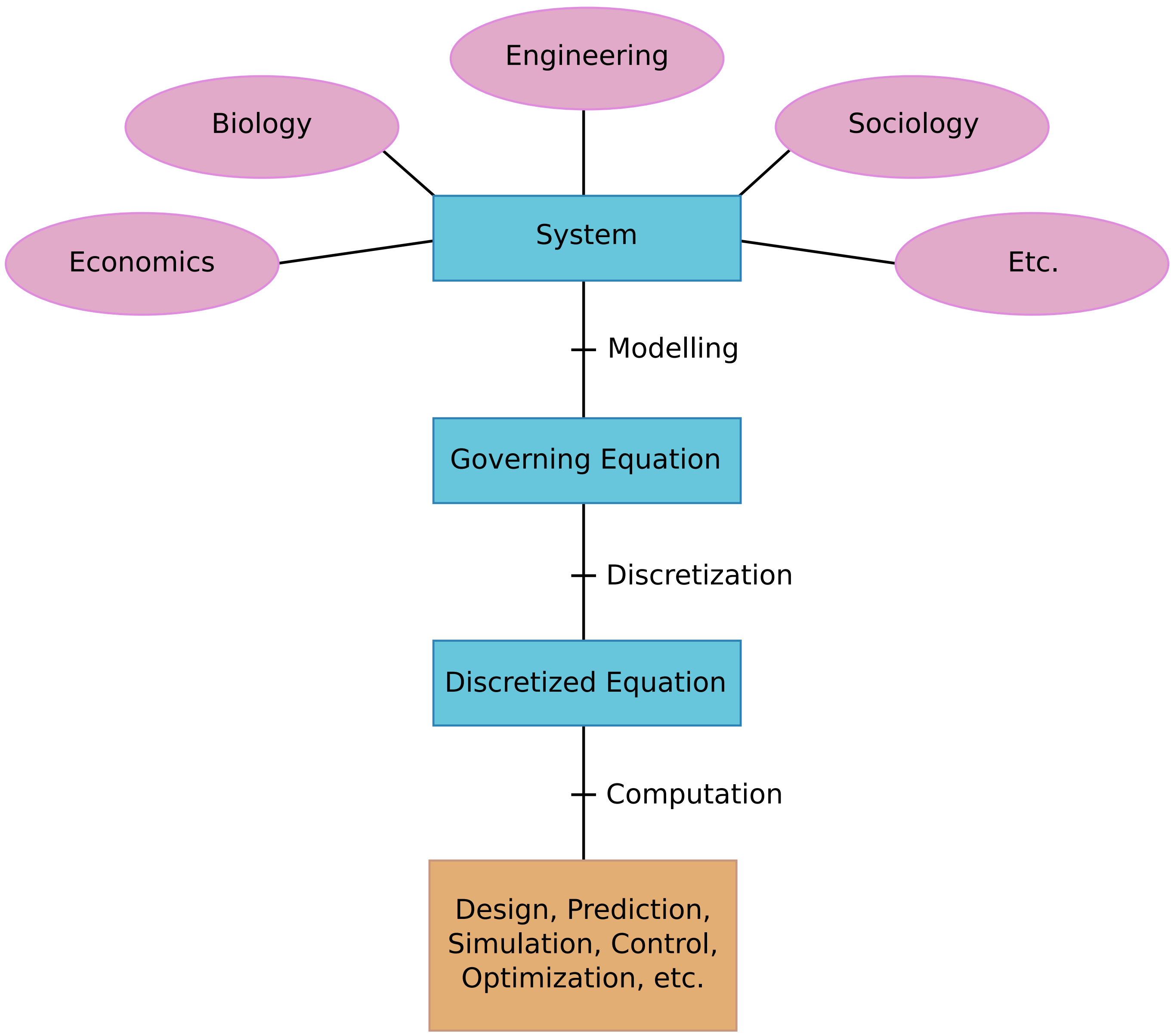
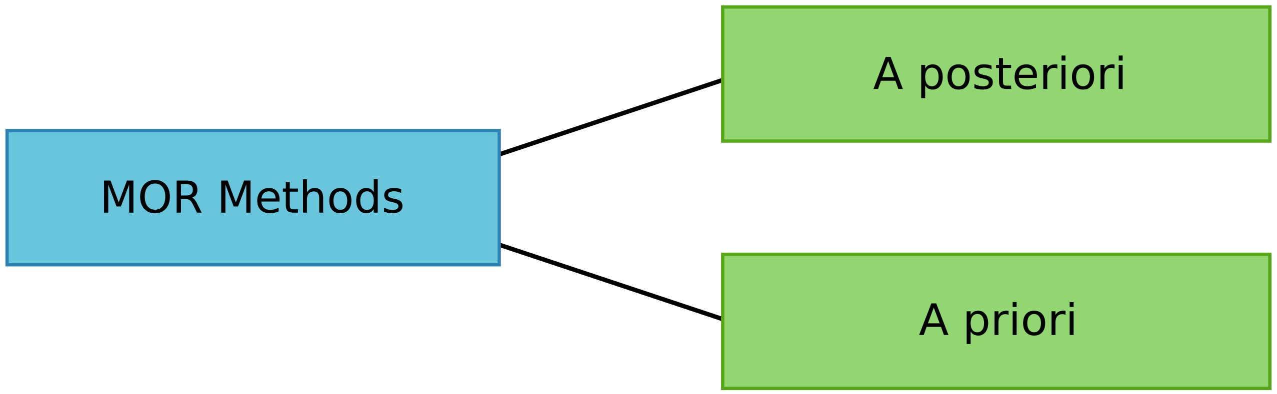


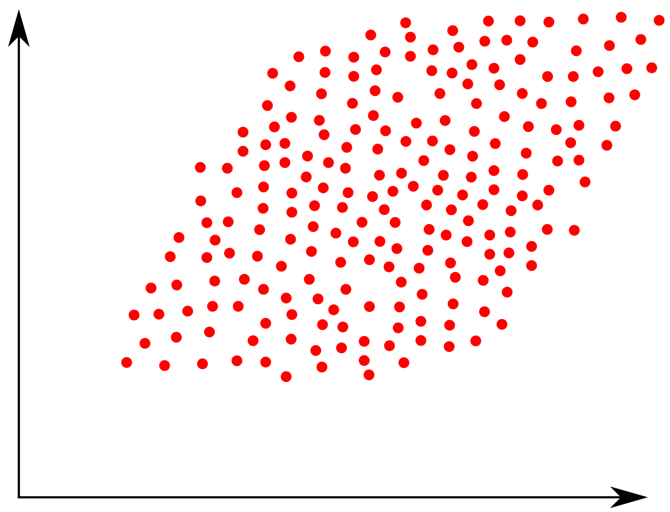
Model Order Reduction (MOR) Methods



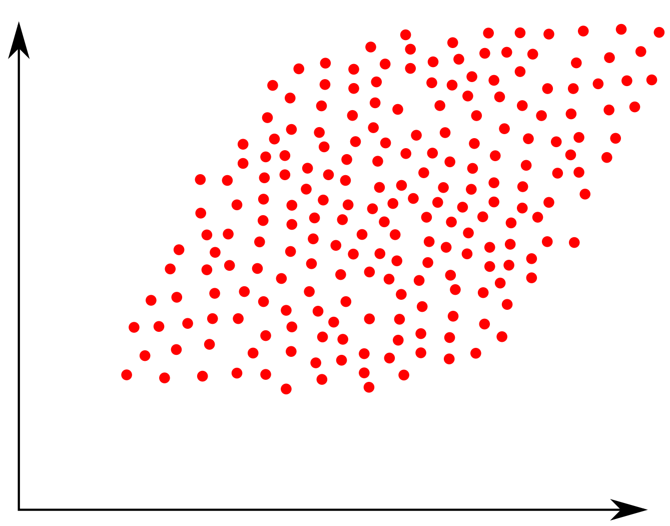
A posteriori methods:

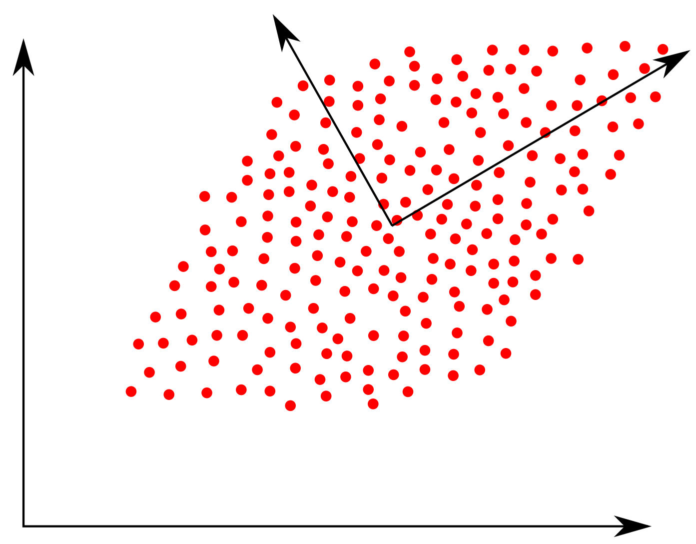
- Built after computing some solutions of the system or the whole set of solutions.
- Especially useful when the model has a relatively small number of parameters but many observations.
- Most famous example is Proper Orthogonal Decomposition (**POD**) based on Principal Component Analysis (**PCA**).
- Reduced Basis Method (**RB**) is another example.

Model Order Reduction (MOR) Methods



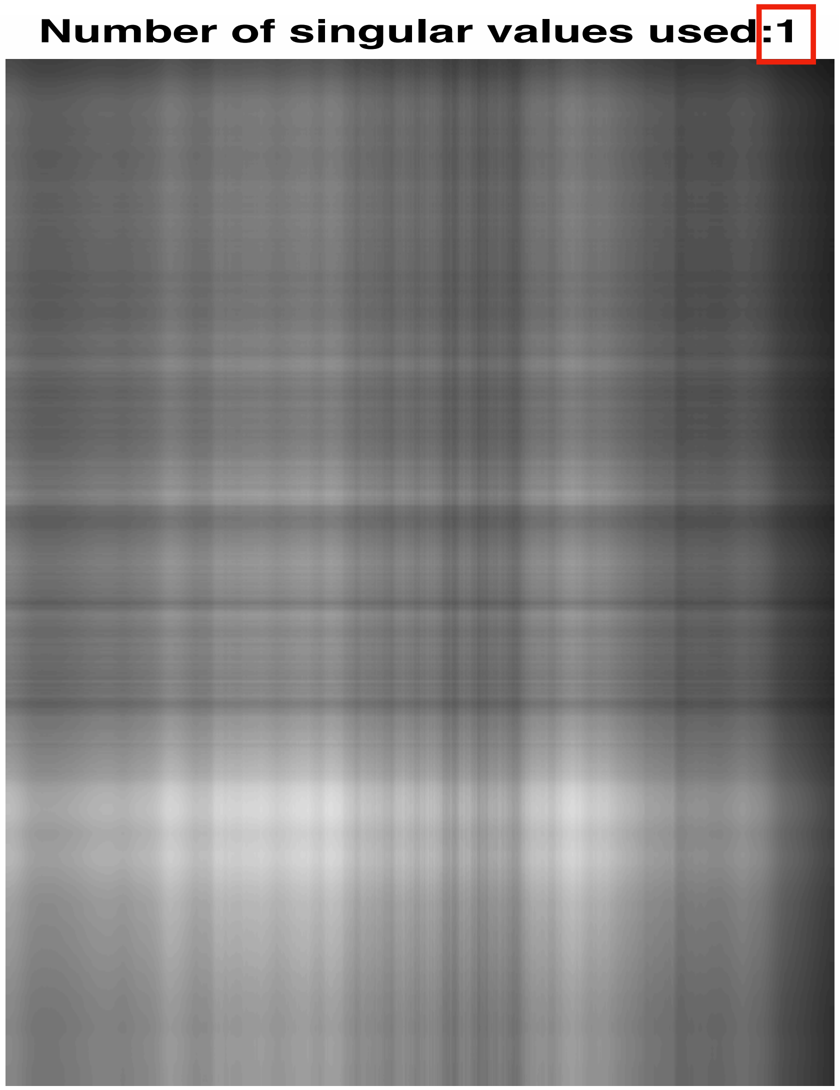
Model Order Reduction (MOR) Methods



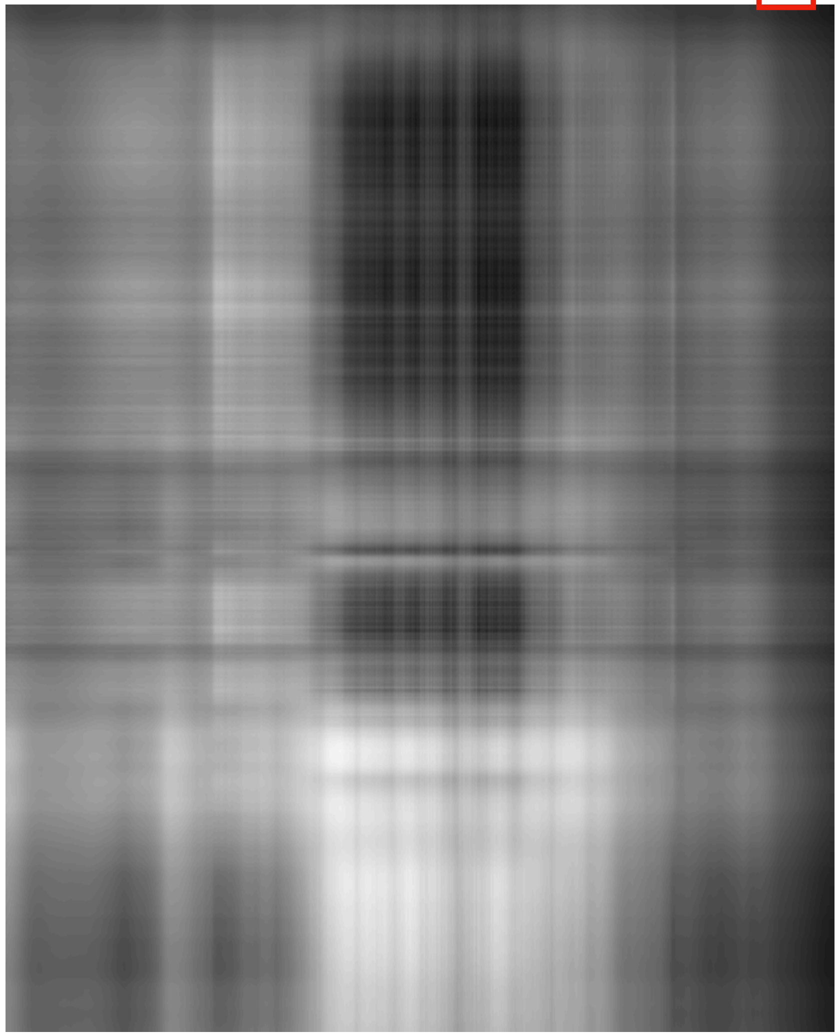


A posteriori methods

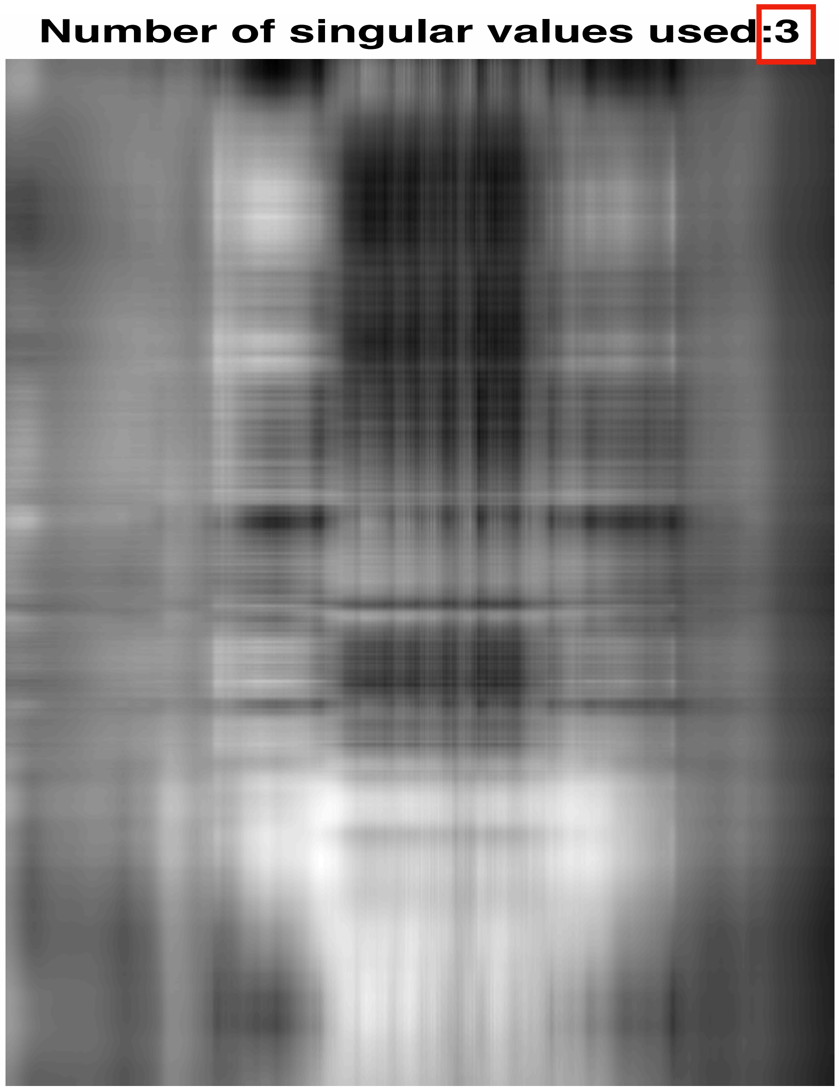
MOR Methods



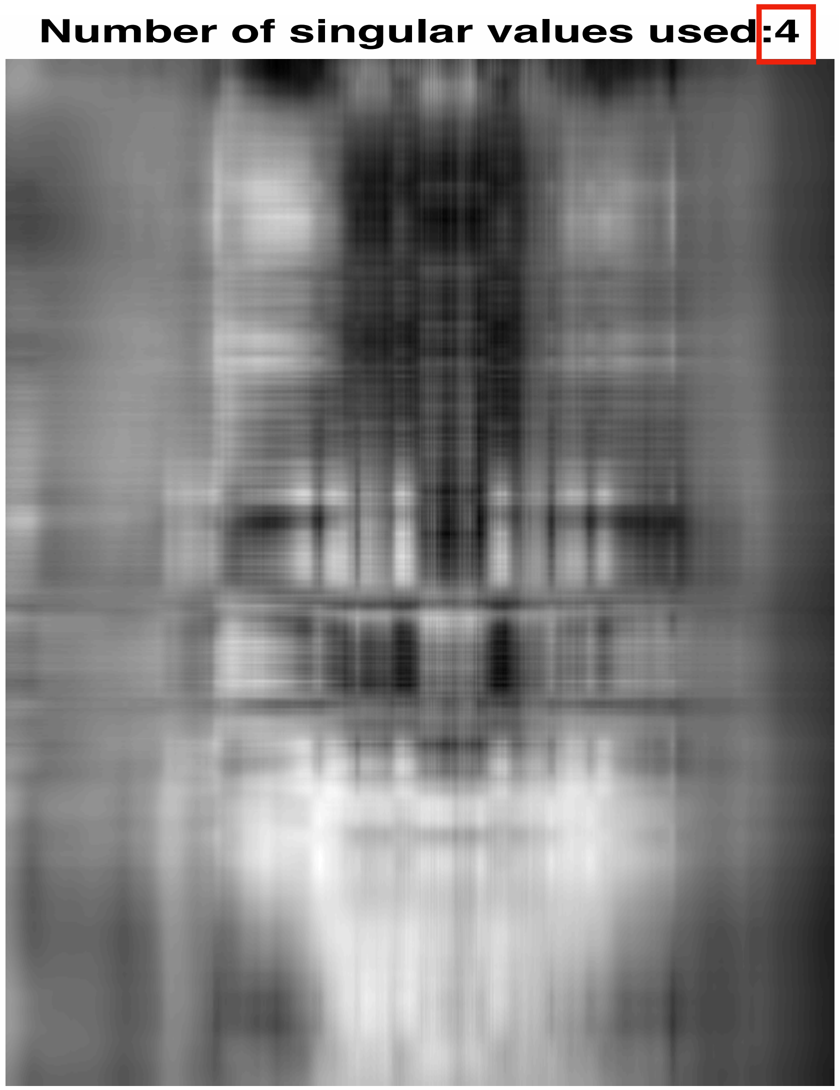
MOR Methods



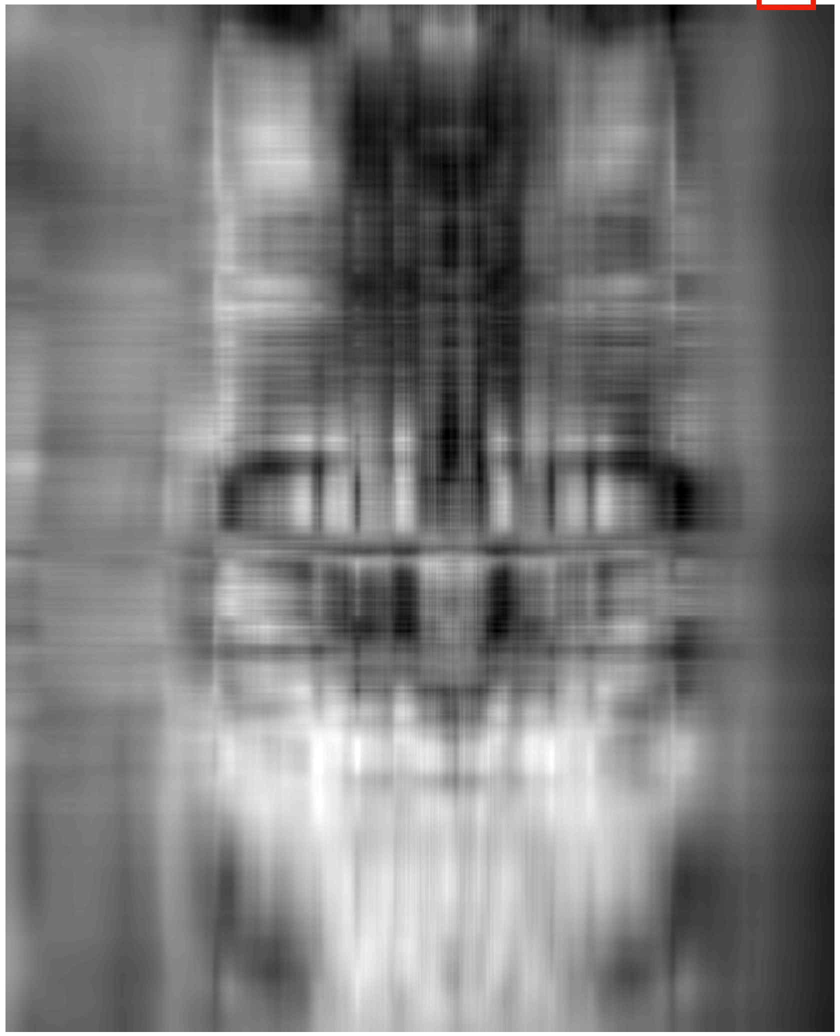
MOR Methods



MOR Methods

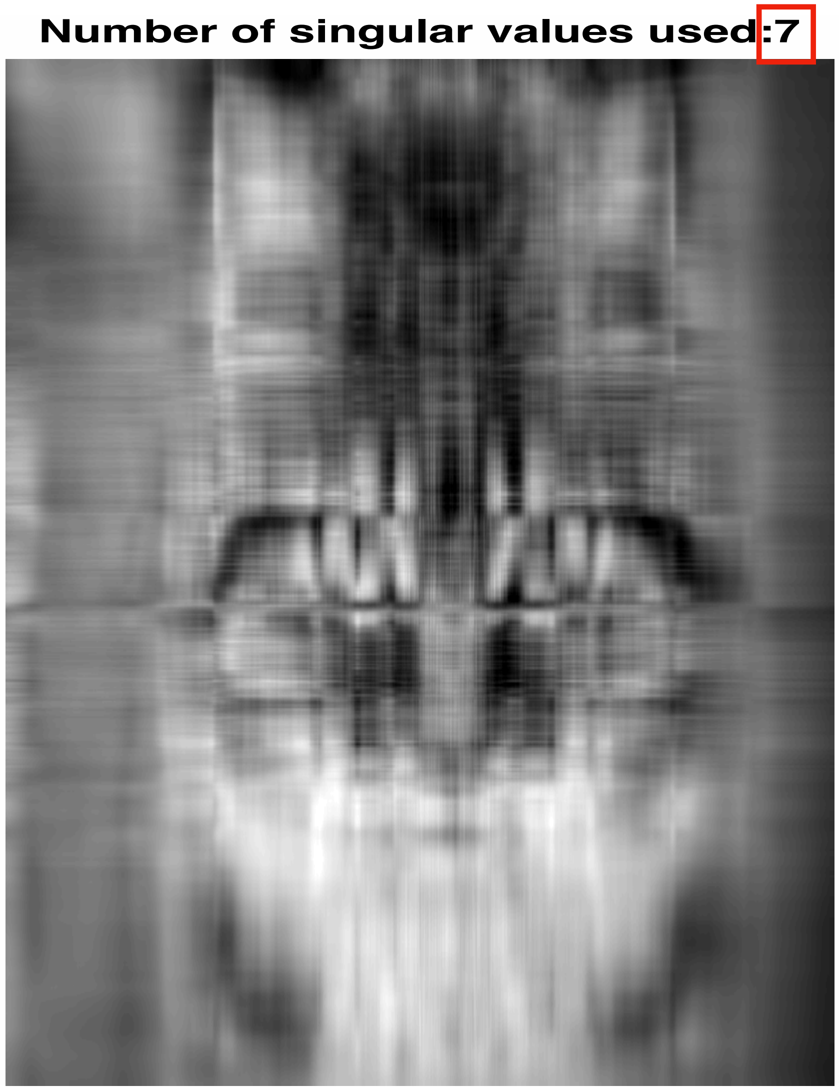


MOR Methods

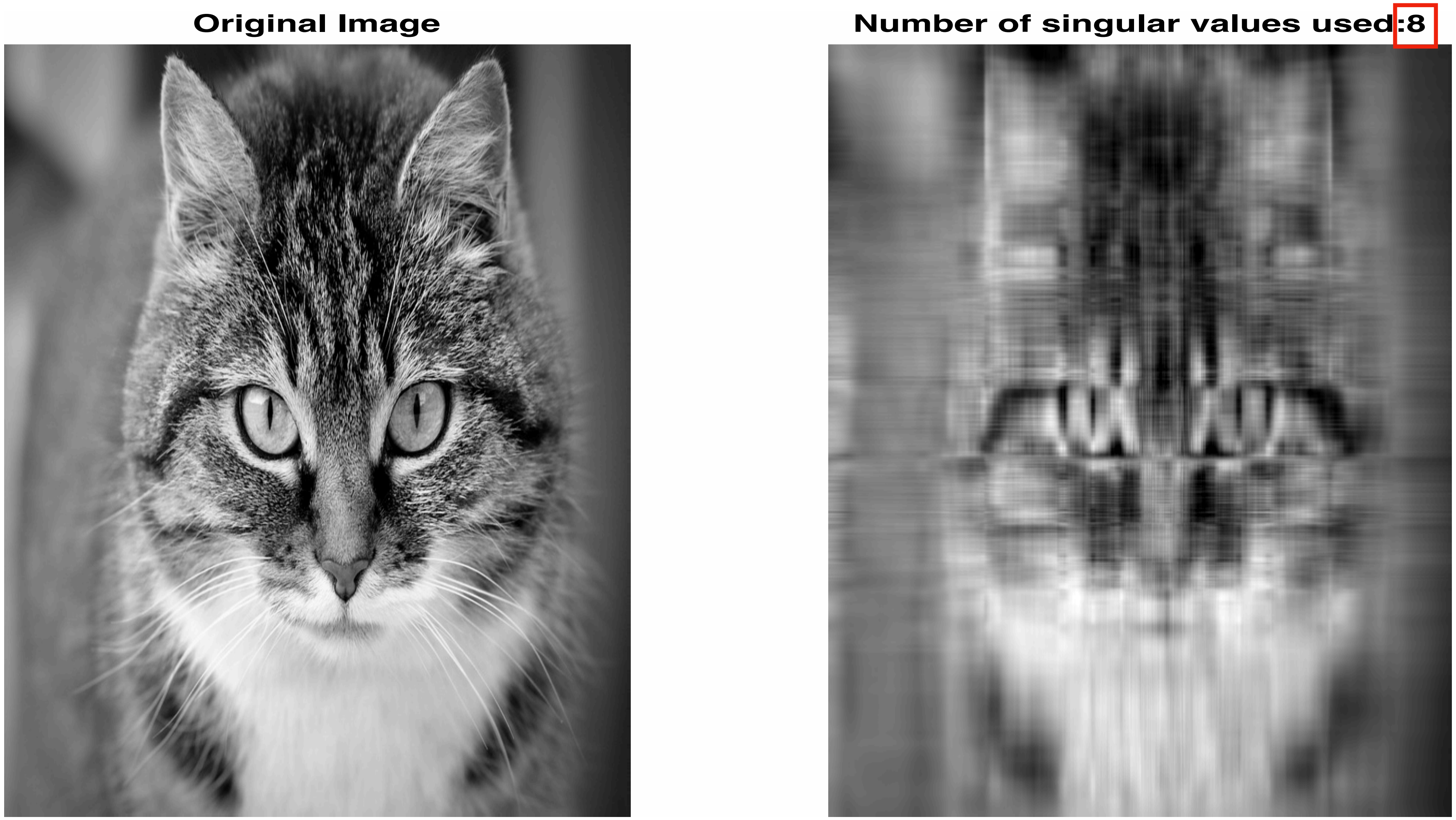


MOR Methods

MOR Methods



MOR Methods

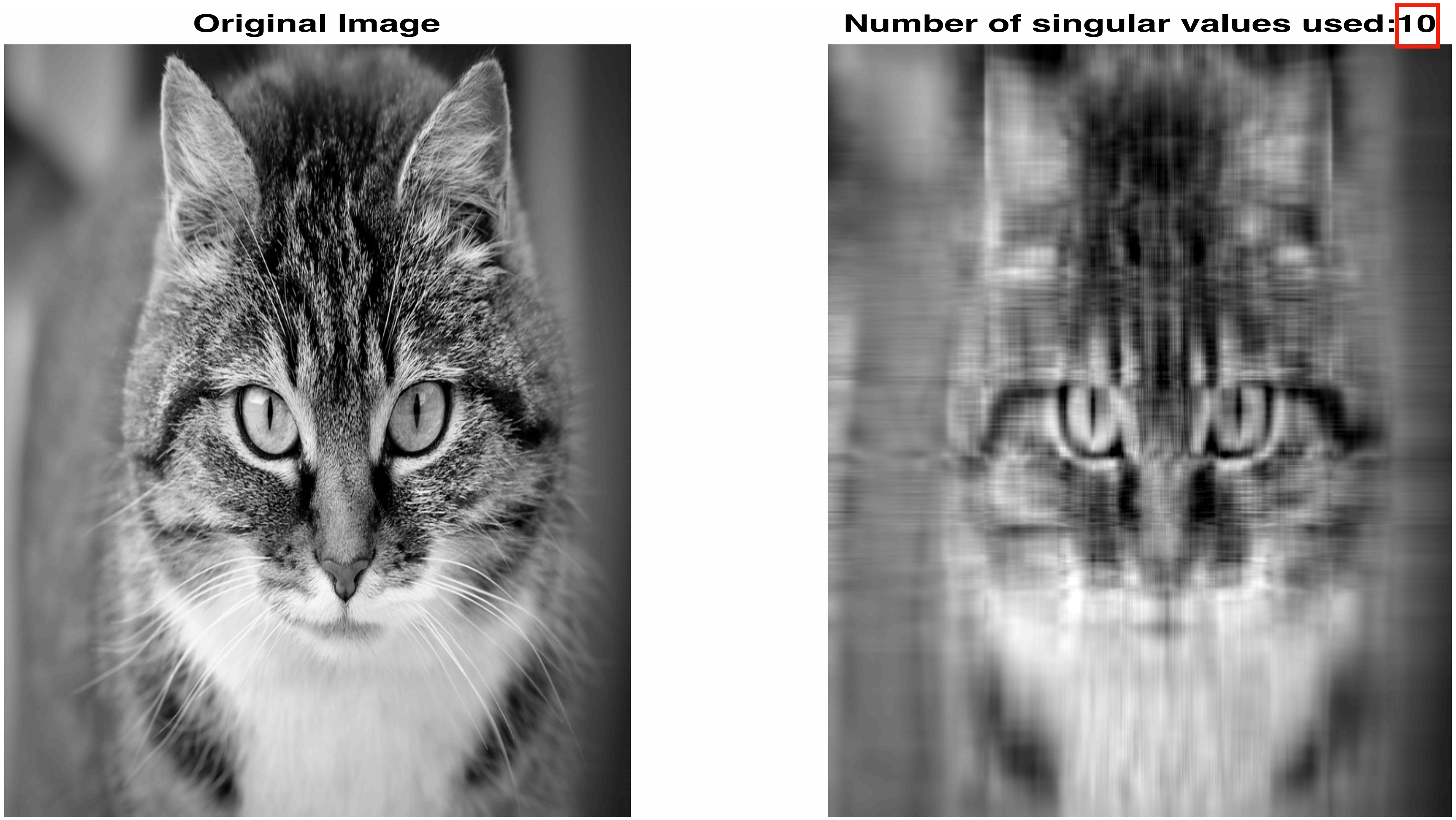


MOR Methods

Original Image

Number of singular values used:9

MOR Methods



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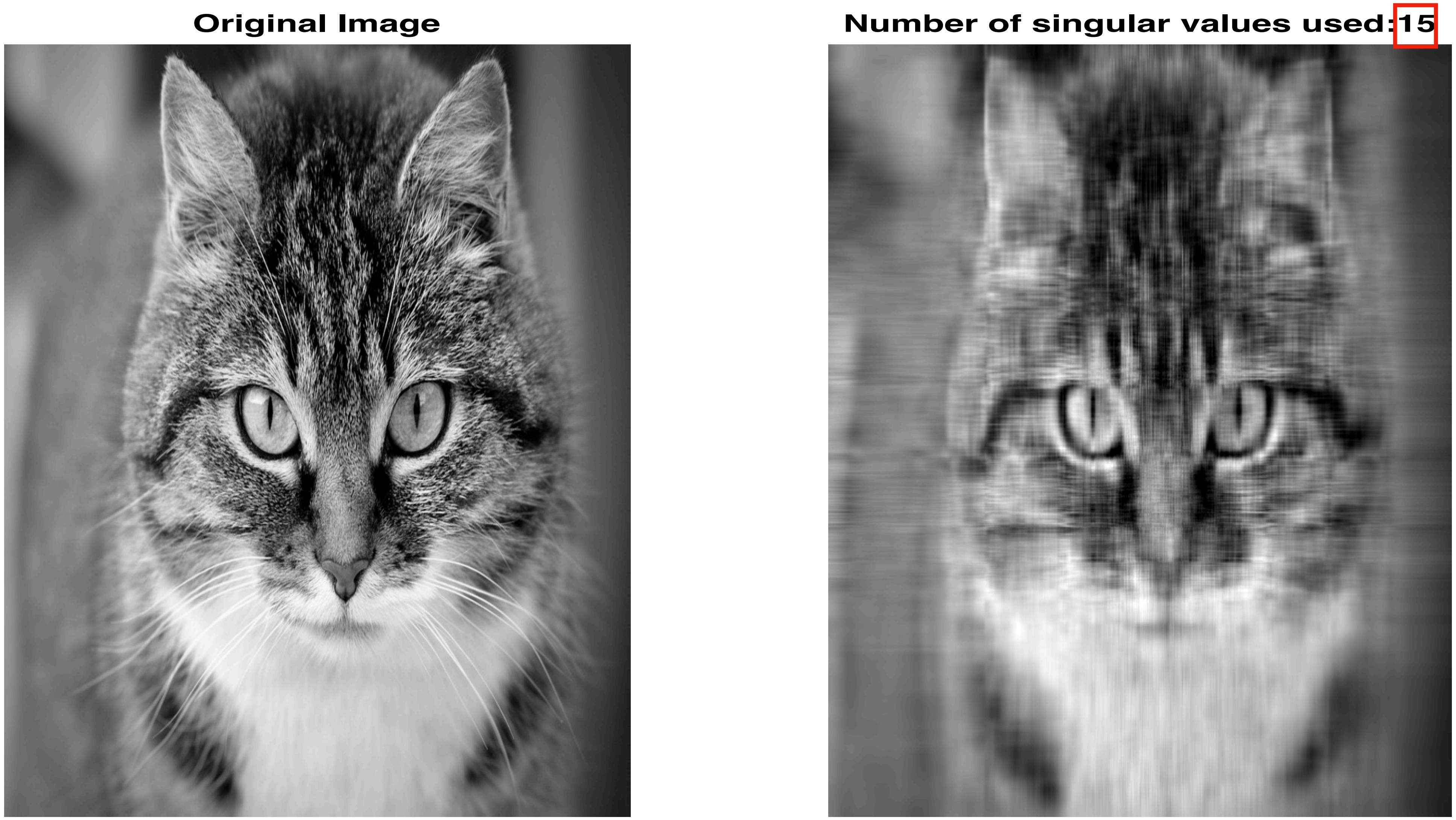

MOR Methods

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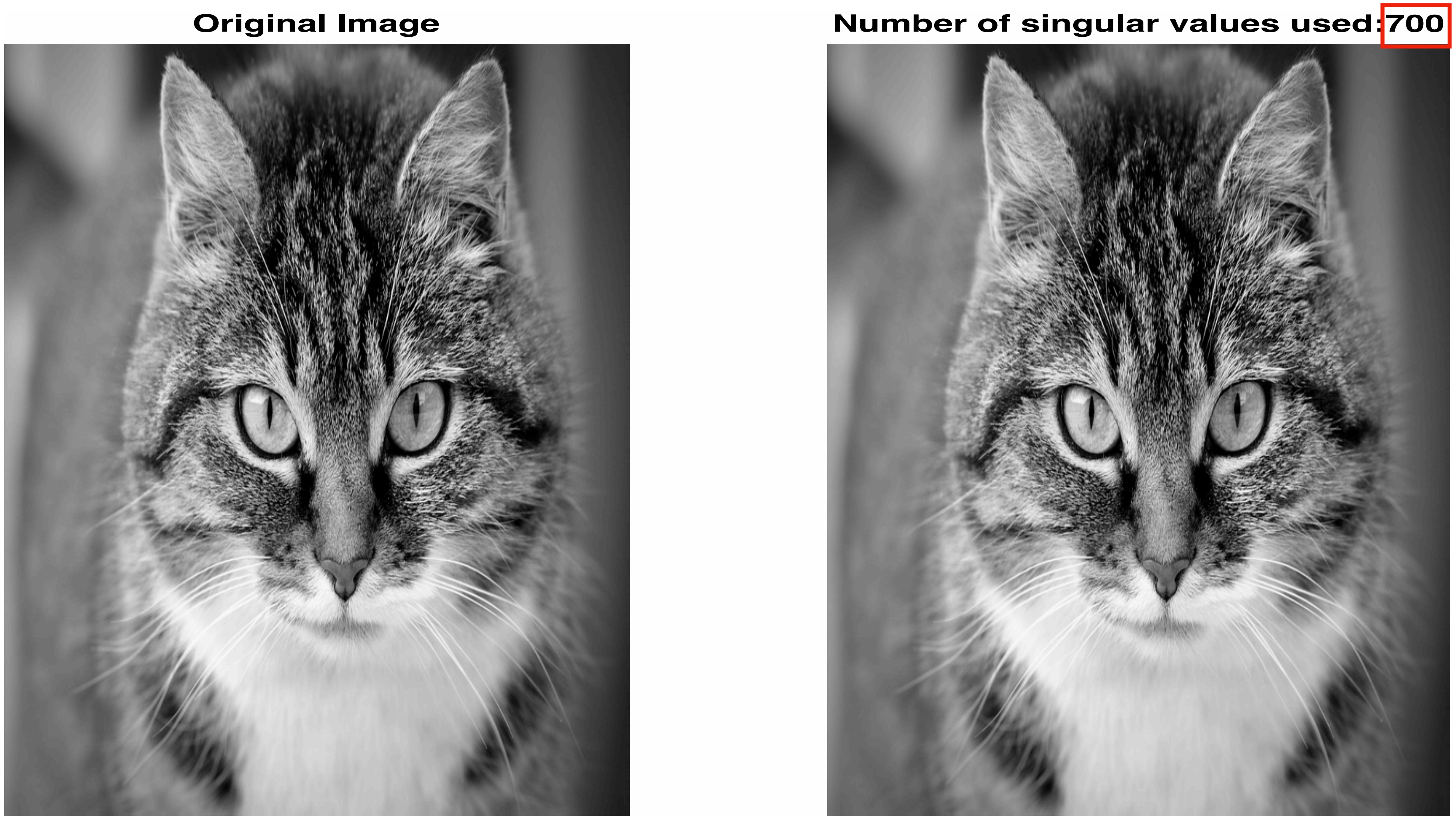

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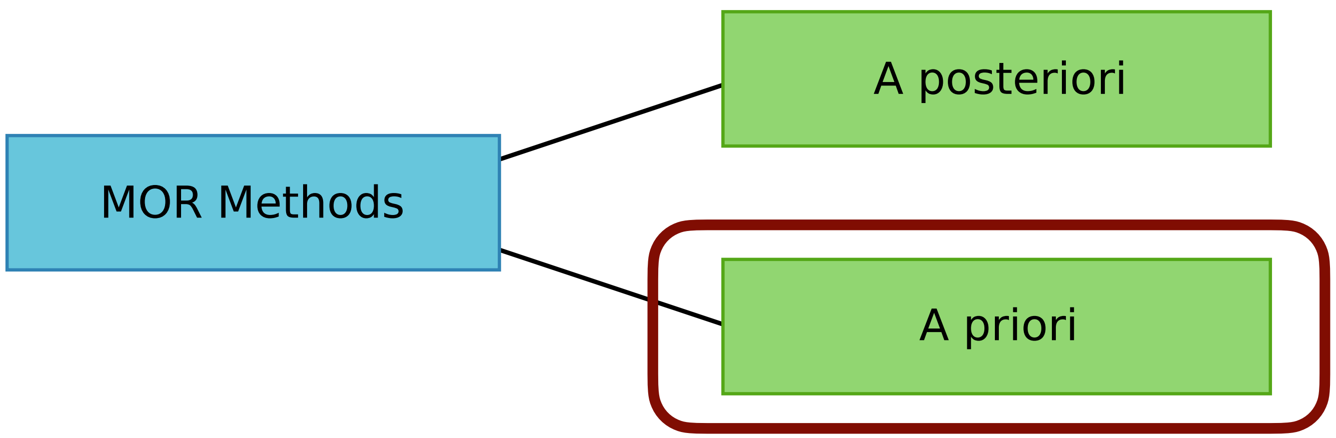

A priori methods:

- Built without the need of pre-computing any solution of the problem.
- Based in the knowledge of the equation that governs the problem.
- A representative example of this kind of methods is the Proper Generalized Decomposition (**PGD**).

Proper Generalized Decomposition (PGD)

Problem stated by its governing differential equation:

$$\mathcal{L}(u) = 0 \ \ in \ \ \Omega$$

Key point: Assumption that the solution of the governing equation can be approximated as a finite sum of products of functions depending only on one variable.

$$\boldsymbol{u}(x_1, x_2, \ldots, x_D) \approx \sum_{i=1}^{N} F_i^1(x_1) \cdot F_i^2(x_2) \cdot \ \ldots \ \cdot F_i^D(x_D)$$

where N is the number of sums (called *modes*) and u is the solution of the equation depending on D independent variables:

- Physical space variables (cartesian coordinates)
- Parameters that modify the behavior (E, ν in linear elasticity).
- ...

MOR Methods

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MOR Methods

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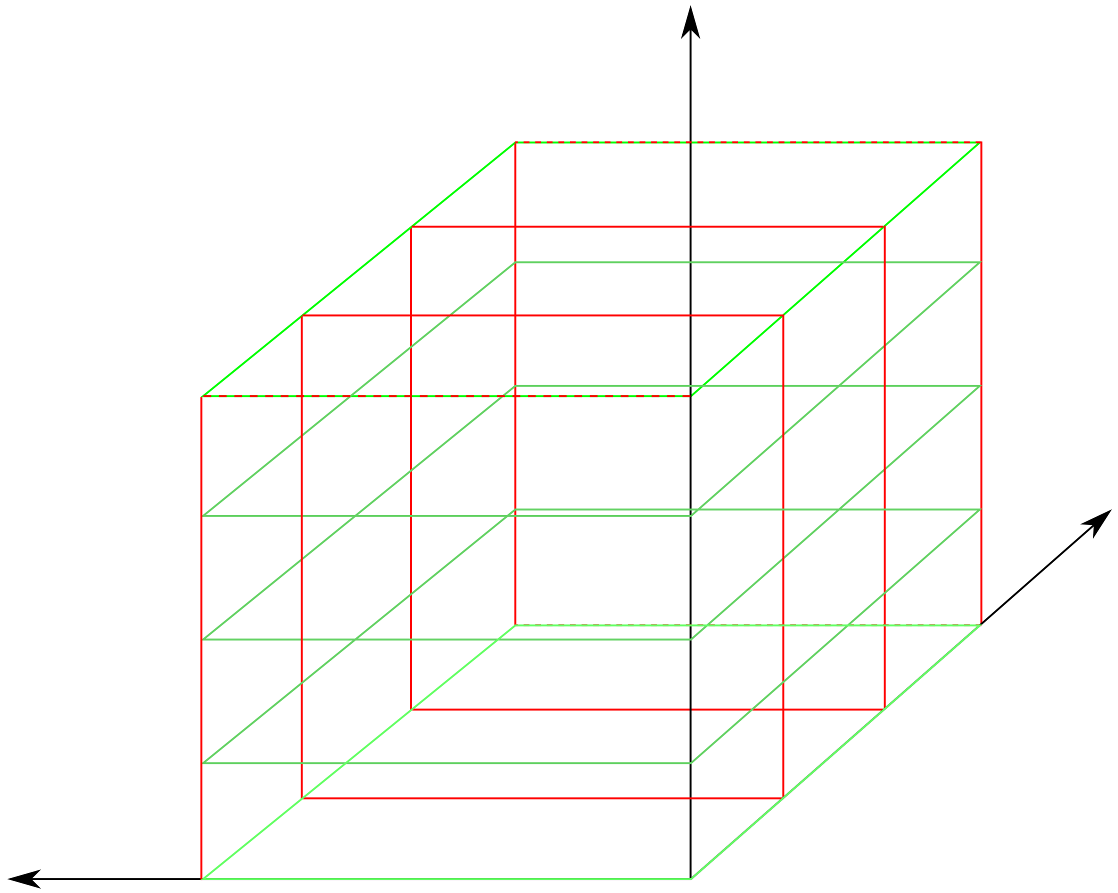

No reduction:

$Dim = M_1 \times M_2 \times M_3$

Separate variables reduction:

$dim = M_1 \times N + M_2 \times N + M_3 \times N$

MOR Methods

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CURSE OF DIMENSIONALITY!

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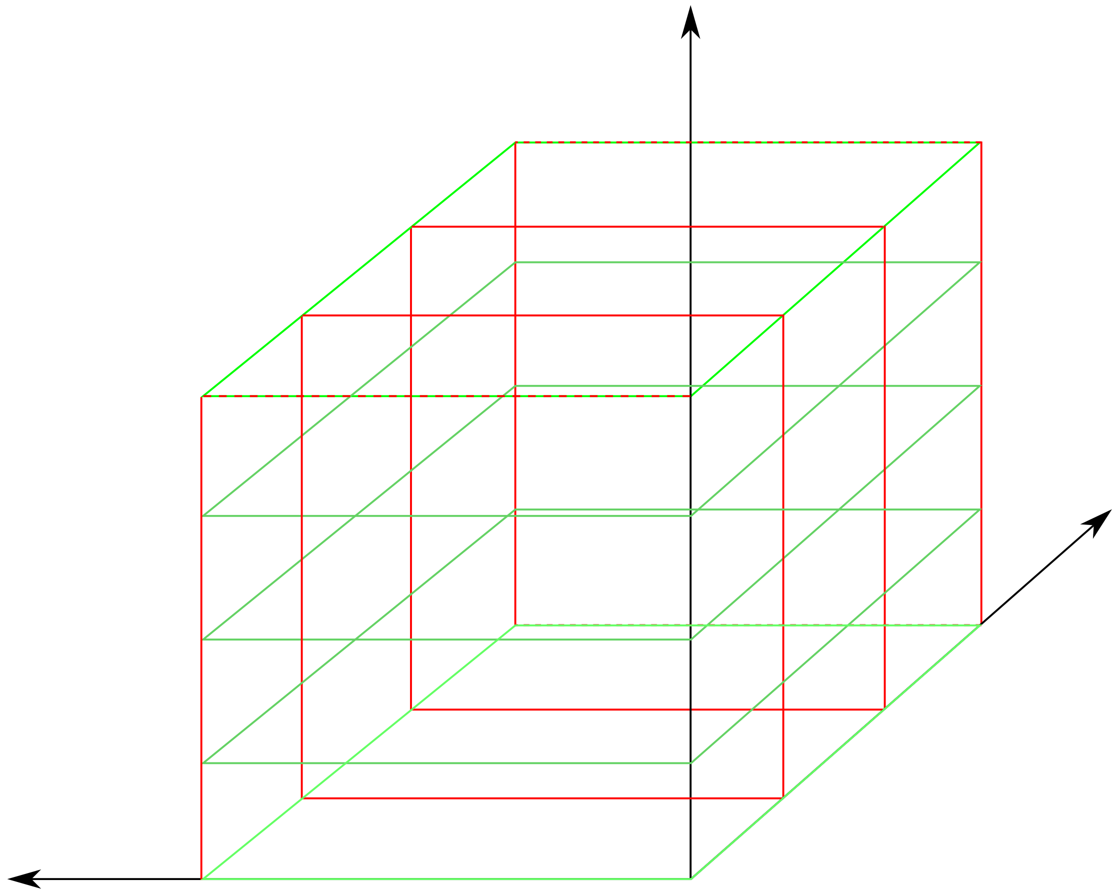

MOR Methods

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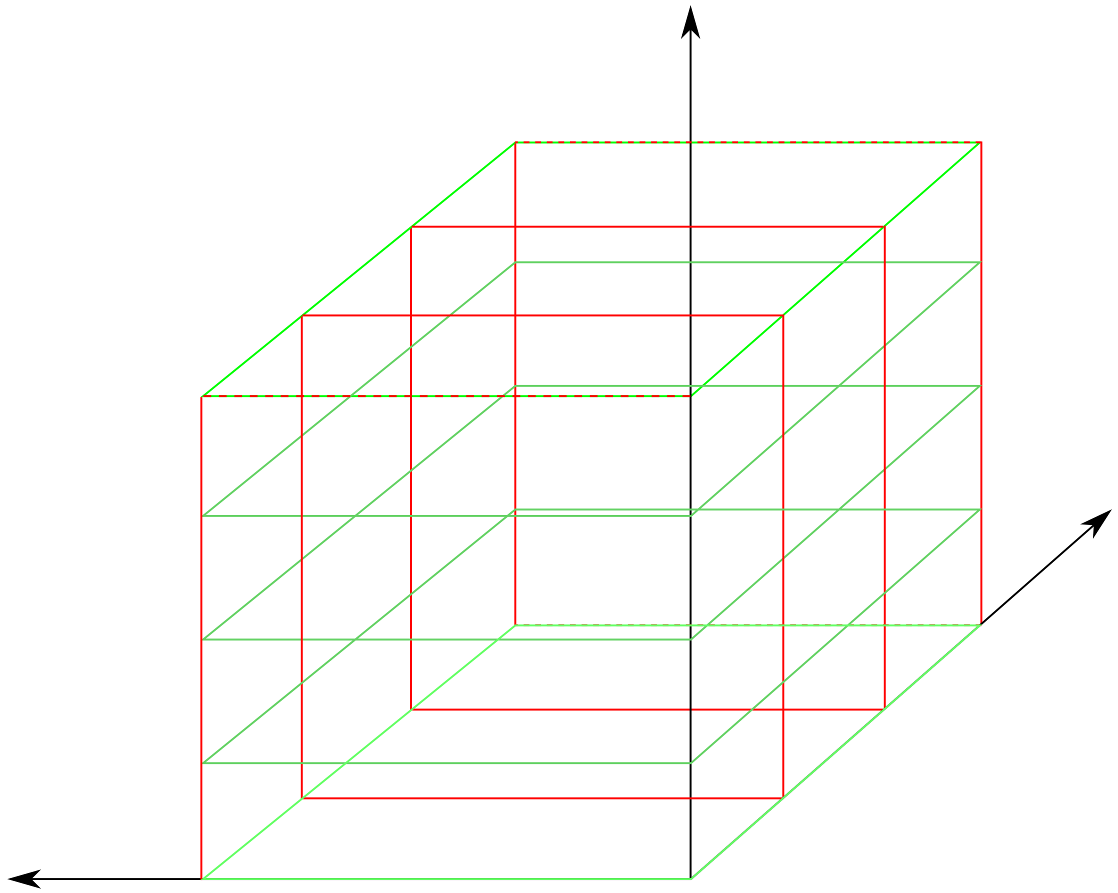
CURSE OF DIMENSIONALITY!



Separate variables reduction:

$dim = M_1 \times N + M_2 \times N + M_3 \times N$

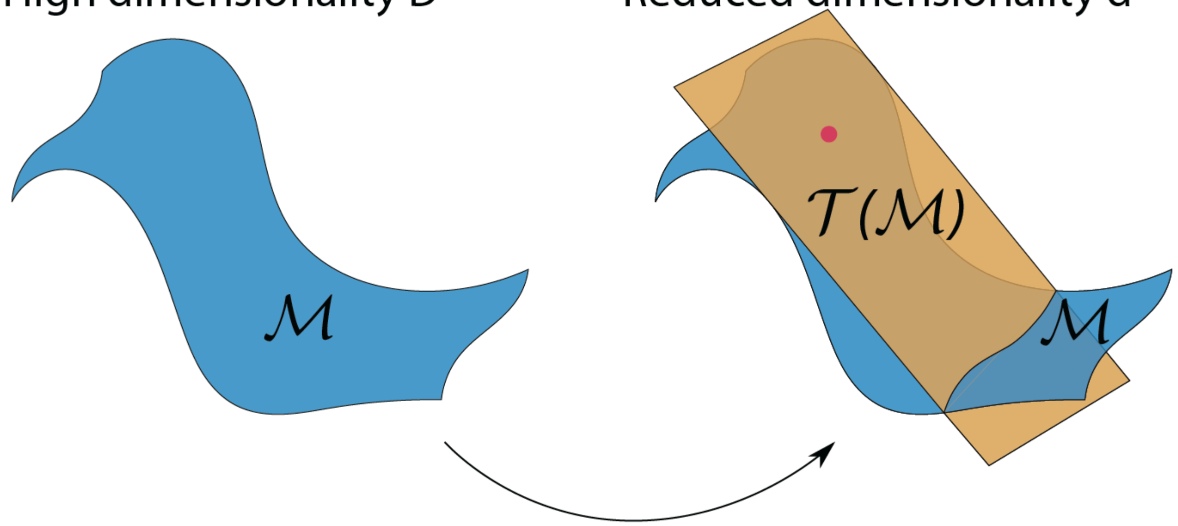
MOR Methods

A posteriori methods

E.g. Proper Orthogonal Decomposition (POD)

Manifold of solutions
High dimensionality D

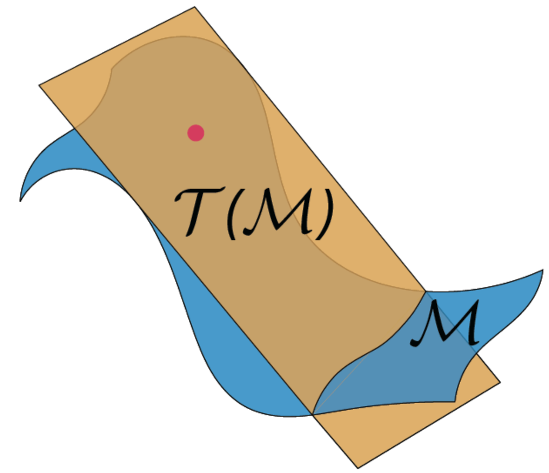
Projected space
Reduced dimensionality d



Dimensionality reduction
D$\gg$d

A priori methods

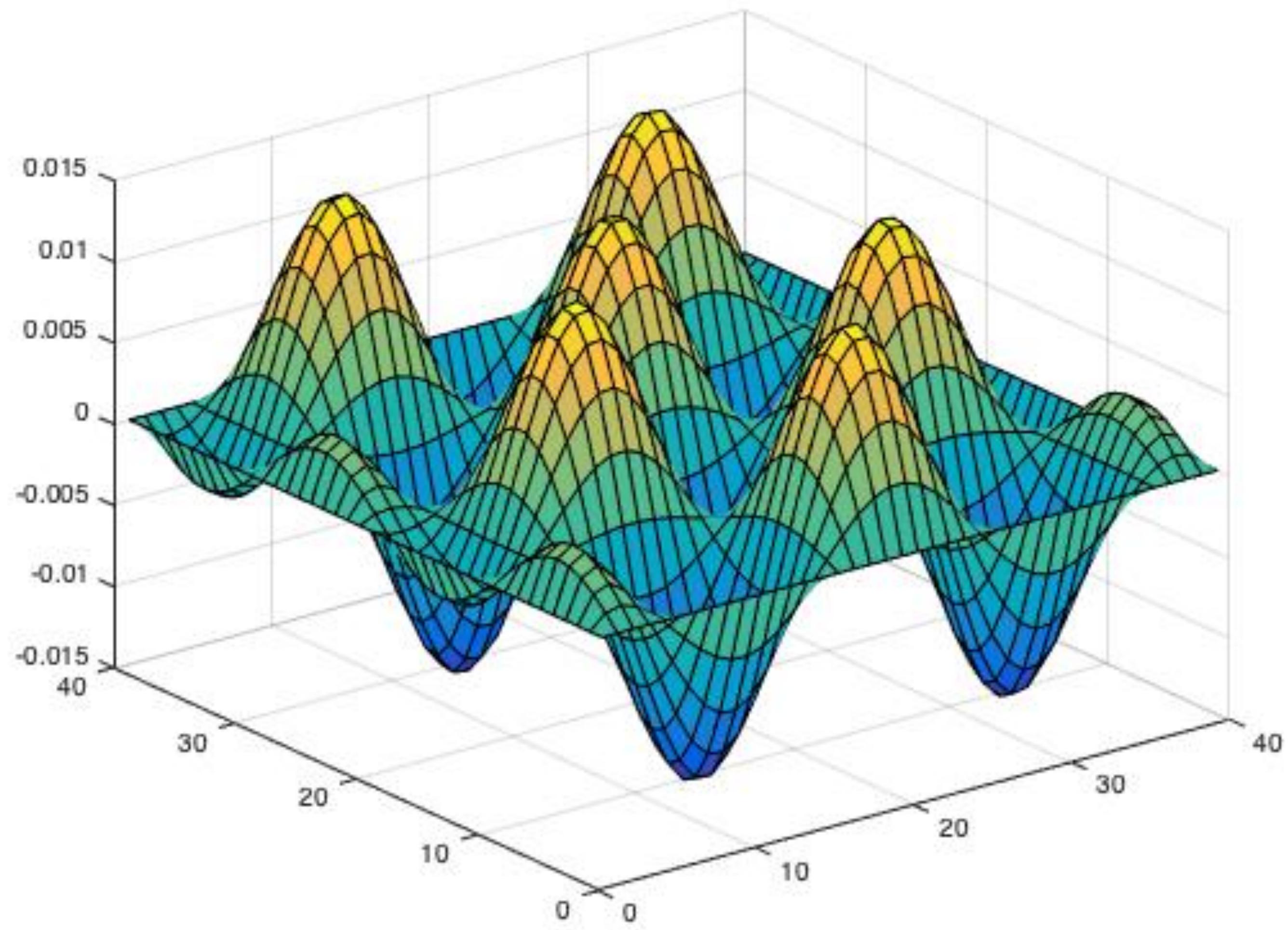
E.g. Proper Generalized Decomposition (PGD)



Solving the solution directly
in the reduced dimensionality d

Proper Generalized Decomposition (PGD)

Example: Poisson problem



Proper Generalized Decomposition (PGD)

Example: Poisson problem

- The strong form of the problem is defined as

$$\triangle u = -f(x_1, x_2, ..., x_D)$$

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$$\int_\Omega \nabla u^* \cdot \nabla u d\Omega = \int_\Omega u^* f \; d\Omega$$

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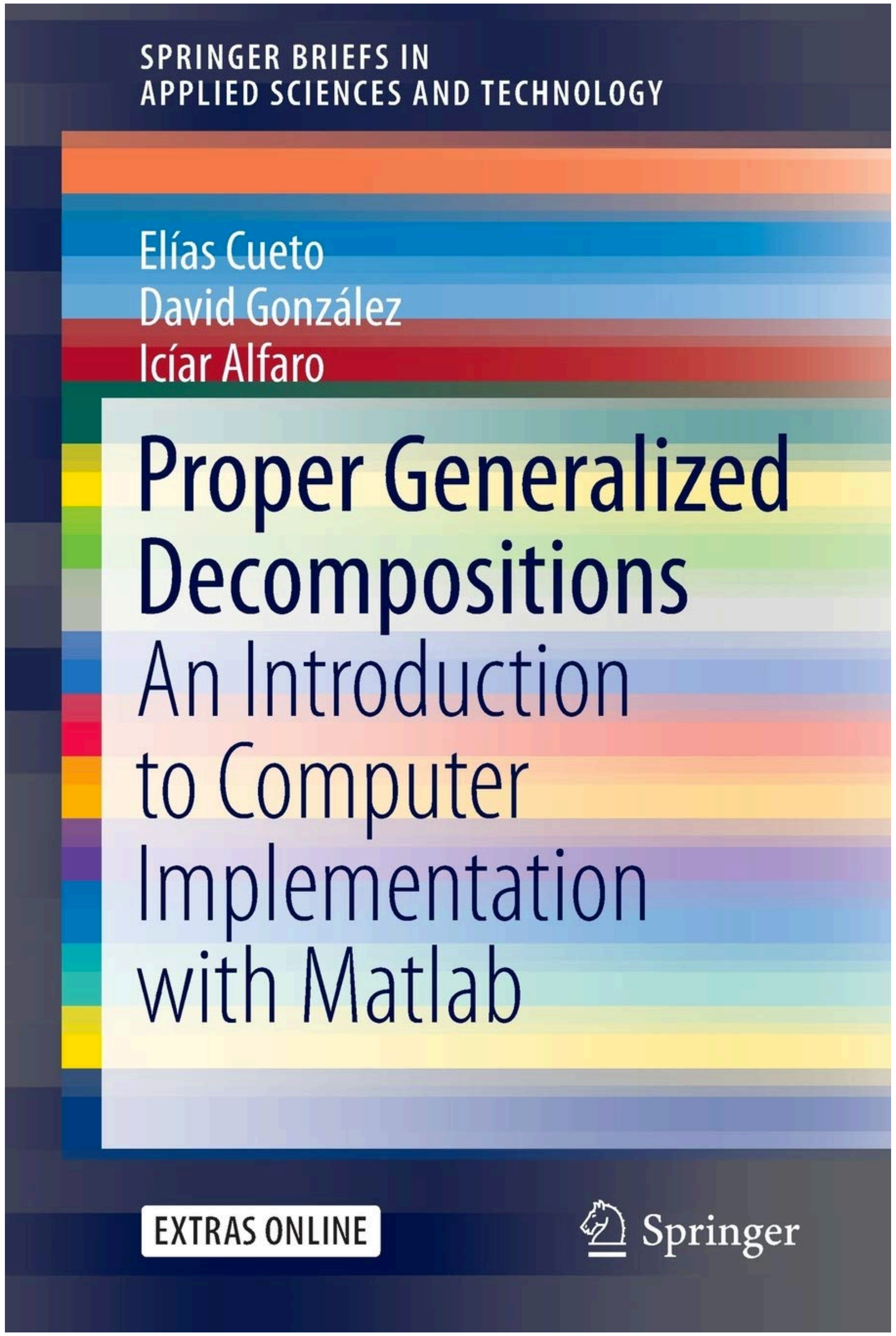
- The weak form is

$$\int_\Omega \nabla u^* \cdot \nabla u d\Omega = \int_\Omega u^* f \; d\Omega$$

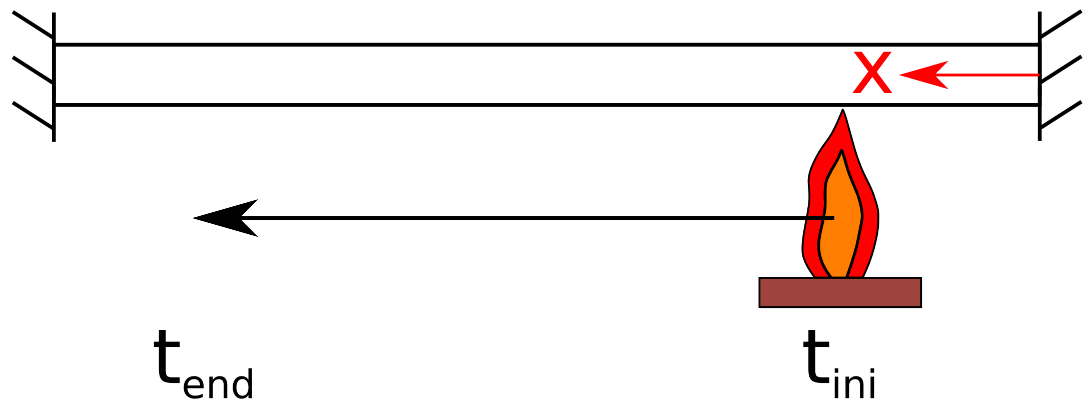
- Test function is chosen as

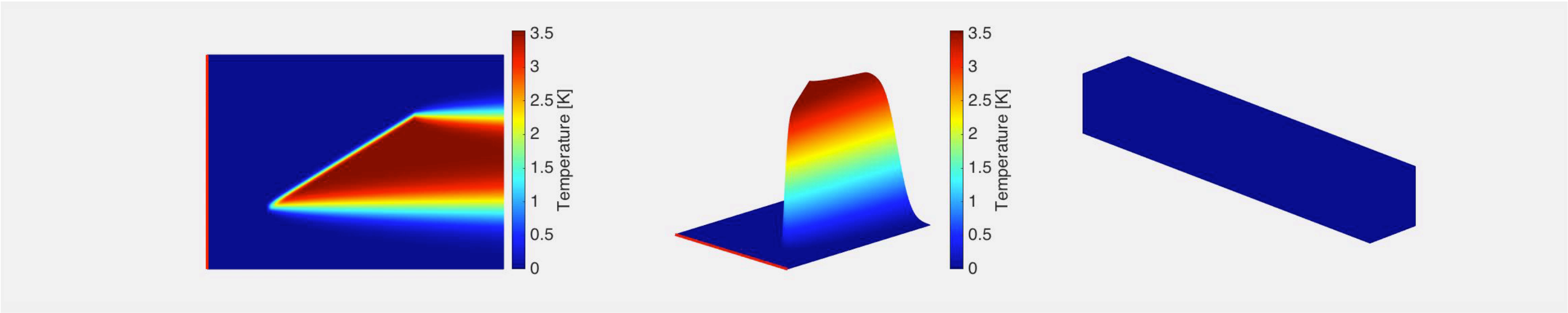
$$\begin{aligned} & u^*(x_1, x_2, \ldots, x_D) \\ = {} & R_1^*(x_1) \cdot R_2(x_2) \cdot \ldots \cdot R_D(x_D) + \ldots + R_1(x_1) \cdot R_2(x_2) \cdot \ldots \cdot R_D^*(x_D) \end{aligned}$$

Cueto, E., González, D., & Alfaro, I. (2016). Proper generalized decompositions: an introduction to computer implementation with Matlab. Springer.



Heat Transient Equation





Governing Equation:

$$u(\boldsymbol{x},t)=\begin{cases}\rho c_p\partial_t u-\nabla\cdot k\nabla u &= f(\boldsymbol{x},t) \quad \text{in } \Omega\times\mathcal{I},\\ u &= u_D \quad \text{on } \Gamma_D\times\mathcal{I},\\ u &= u_0 \quad \text{on } \Omega\times\{0\}.\end{cases}$$

Heat Transient Equation

$$\boxed{\rho c_p \frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)}$$

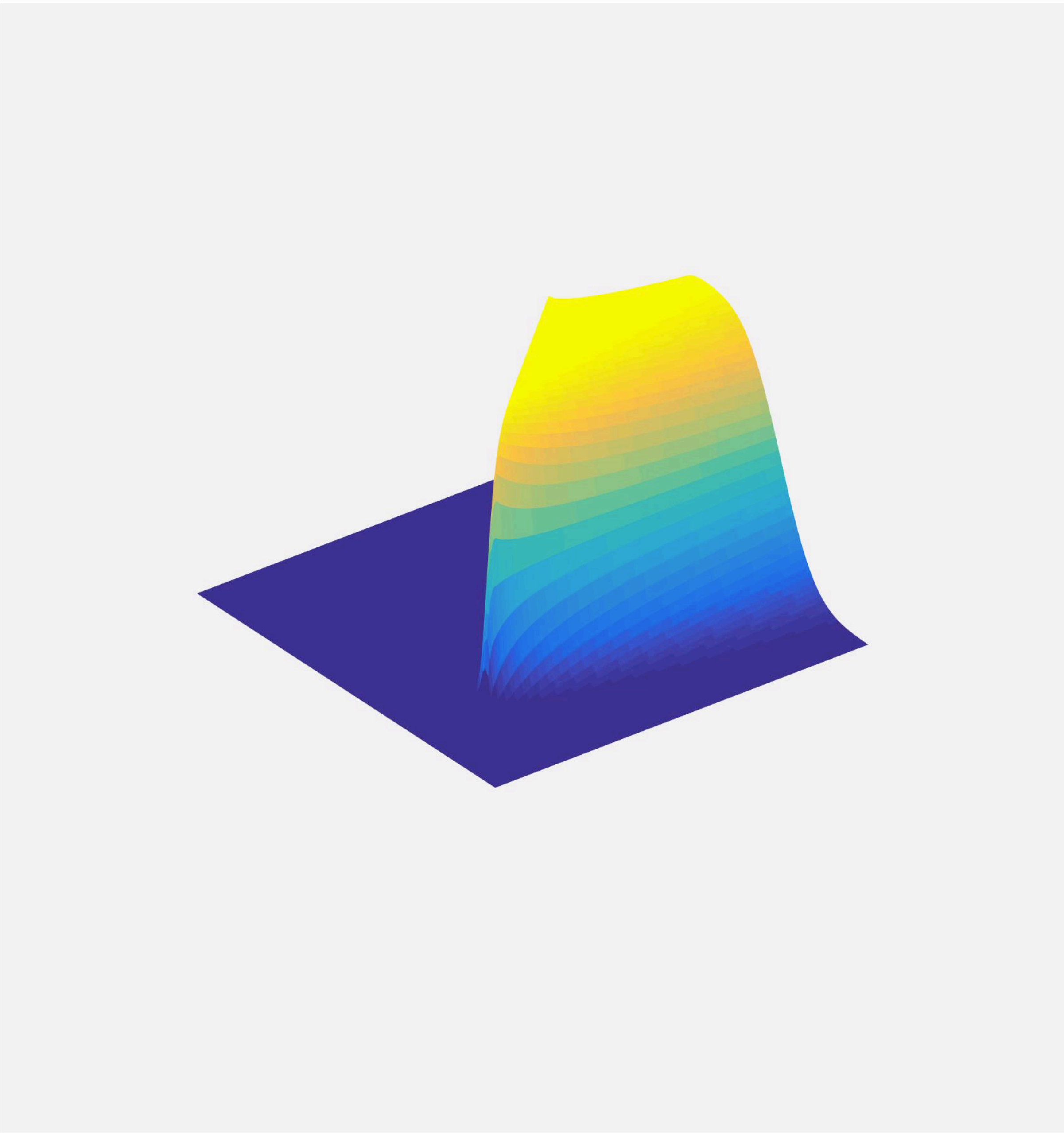
Structure of the solution as a sum of n products of separate functions $F_i(x)$ and $G_i(t)$

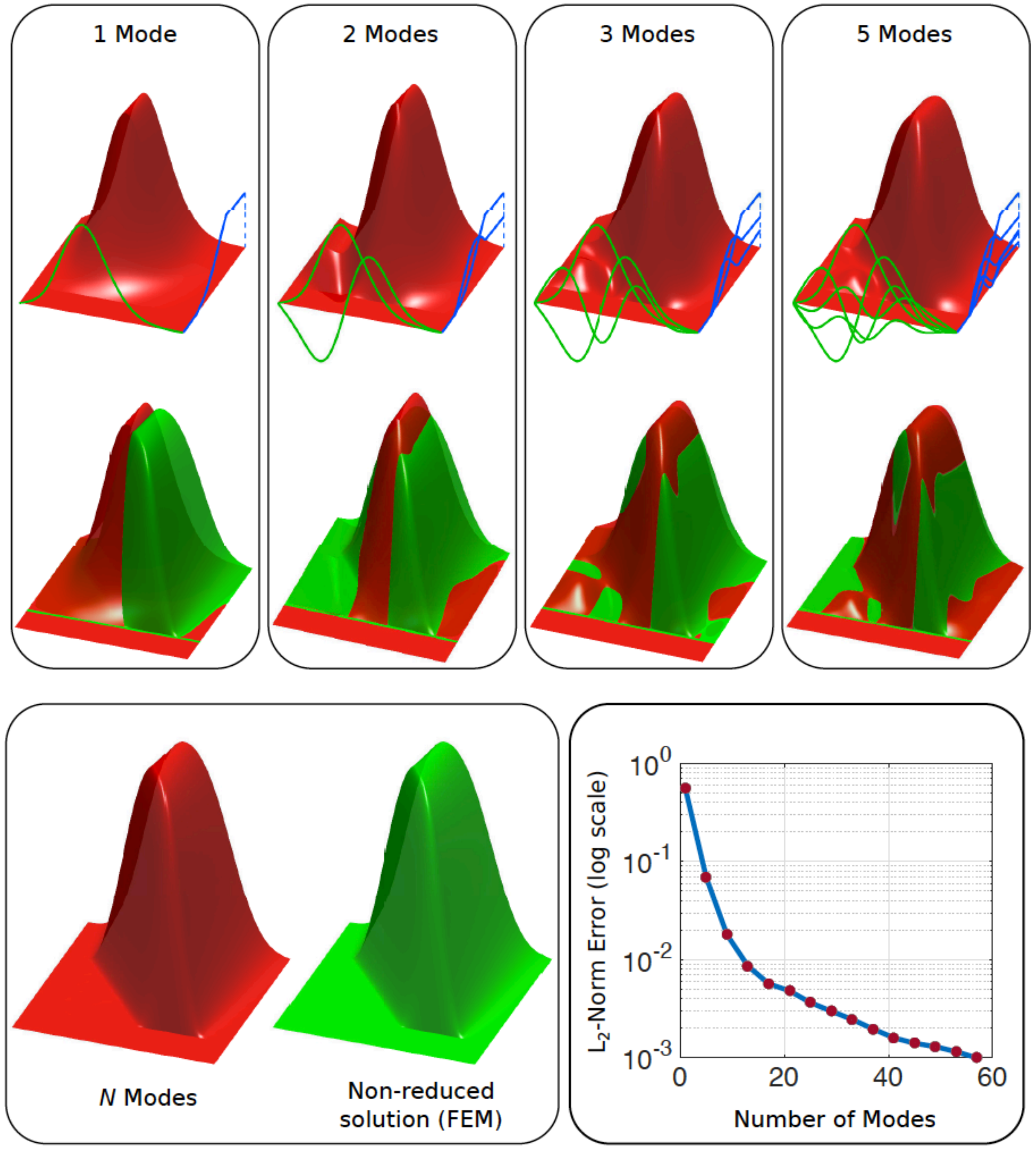
$$u(x,t) \approx \sum_{i=1}^{n} [F_i(x) \cdot G_i(t)]$$

The source term is decompose in the sum of m products of separate variables

$$f(x,t) \approx \sum_{h=1}^{m} fa_h(x) \cdot fb_h(t)$$

Heat Transient Equation





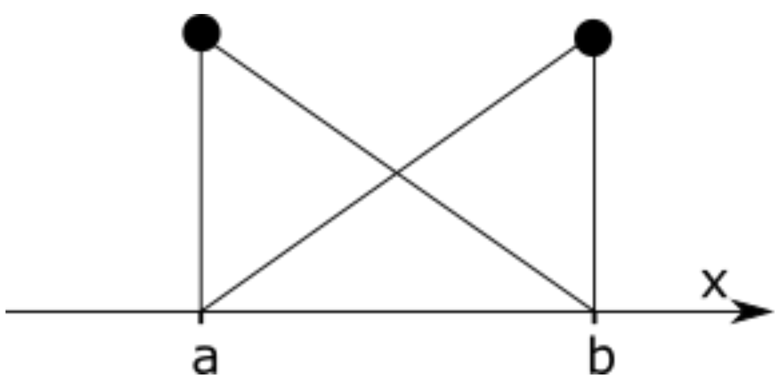
Heat Transient Equation

PGD applied over Finite Element Methods

Linear Function approximation

$$N_1 = \left[\frac{b-x}{b-a}\right]$$

$$N_2 = \left[\frac{x-a}{b-a}\right]$$



Heat Transient Equation

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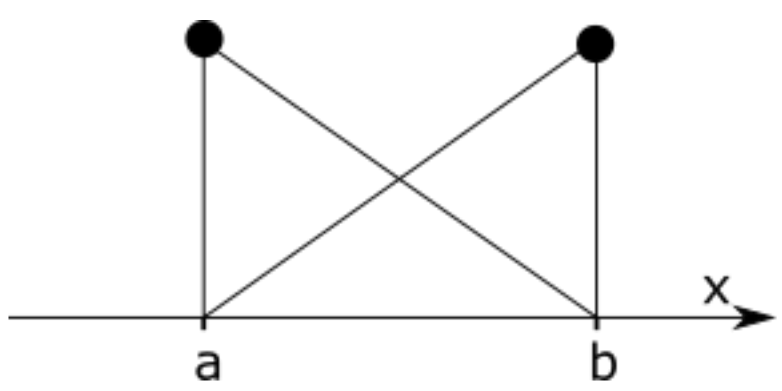

Elemental function

$$T^e = \underline{\mathrm{N}}^{e^T} \underline{\mathrm{T}}^e = N_1 T_1 + N_2 T_2$$

$$\underline{\mathrm{N}}^{e^T} = [N_1 \; N_2] \qquad \underline{\mathrm{T}}^e = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

Heat Transient Equation

Galerkin projection

$$\int_{\Omega} \omega_k L(\bar{T}) d\Omega = 0$$

Heat Transient Equation

Galerkin projection

$$\int_{\Omega} \omega_k L(\bar{T}) d\Omega = 0$$

Residual weights applied to our equation:

$$\int_{\Omega} \rho c_p \omega_k \frac{\partial \underline{\mathrm{T}}}{\partial t} d\Omega - \int_{\Omega} \omega_k \frac{\partial}{\partial x} \left[k_x \frac{\partial \underline{\mathrm{T}}}{\partial x} \right] d\Omega = \int_{\Omega} \omega_k \underline{\mathrm{f}} d\Omega$$

Heat Transient Equation

Galerkin projection

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Using same approximation functions for weights:

$$\int_\Omega \rho c_p N_i \frac{\partial \underline{\mathrm{T}}}{\partial t} d\Omega - \int_\Omega N_i \frac{\partial}{\partial x}\left[k_x \frac{\partial \underline{\mathrm{T}}}{\partial x}\right] d\Omega = \int_\Omega N_i \underline{\mathrm{f}} d\Omega$$

Heat Transient Equation

$$\int_{\Omega} \rho c_p N_i \frac{\partial \underline{\mathrm{T}}}{\partial t} d\Omega - \int_{\Omega} N_i \frac{\partial}{\partial x}\left[k_x \frac{\partial \underline{\mathrm{T}}}{\partial x}\right] d\Omega = \int_{\Omega} N_i \underline{\mathrm{f}} d\Omega$$

Heat Transient Equation

$$\int_\Omega \rho c_p N_i \frac{\partial \underline{T}}{\partial t} d\Omega - \boxed{\int_\Omega N_i \frac{\partial}{\partial x}\left[k_x \frac{\partial \underline{T}}{\partial x}\right] d\Omega} = \int_\Omega N_i \underline{f} d\Omega$$

Applying Green's Theorem:

$$\int_\Omega N_i \frac{\partial}{\partial x}\left[k_x \frac{\partial \underline{T}}{\partial x}\right] d\Omega = k_x N_i \frac{\partial \underline{T}}{\partial x}\bigg|_\Gamma - k_x \int_\Omega \frac{\partial N_i}{\partial x} \frac{\partial \underline{T}}{\partial x} d\Omega$$

Heat Transient Equation

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Remember our boundary and initial conditions:

$$T(x,0) = 0, \qquad T(0,t) = T(\pi,t) \approx 0$$

Heat Transient Equation

$$\int_\Omega \rho c_p N_i \frac{\partial \underline{\mathrm{T}}}{\partial t} d\Omega - \boxed{\int_\Omega N_i \frac{\partial}{\partial x}\left[k_x \frac{\partial \underline{\mathrm{T}}}{\partial x}\right] d\Omega} = \int_\Omega N_i \underline{\mathrm{f}} d\Omega$$

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Heat Transient Equation

Smoothed equation after applying Green:

$$\rho c_p \int_\Omega N_i \frac{\partial \underline{\mathrm{T}}}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial N_i}{\partial x} \frac{\partial \underline{\mathrm{T}}}{\partial x} d\Omega = \int_\Omega N_i \underline{\mathrm{f}} d\Omega$$

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Polynomial functions to interpolate the solution:

$$\rho c_p \int_\Omega N_i N_j \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} T d\Omega = \int_\Omega N_i N_j f d\Omega$$

Heat Transient Equation

Matrix definition (for clarity):

$$[C] = \rho c_p \int_\Omega N_i N_j d\Omega$$

$$[K] = k_x \int_\Omega \frac{dN_i}{dx} \frac{dN_j}{dx} d\Omega$$

$$[F] = \int_\Omega N_i N_j f d\Omega$$

Heat Transient Equation

Matrix definition (for clarity):

$$[C] = \rho c_p \int_\Omega N_i N_j d\Omega$$

$$[K] = k_x \int_\Omega \frac{dN_i}{dx} \frac{dN_j}{dx} d\Omega$$

$$[F] = \int_\Omega N_i N_j f d\Omega$$

Final FEM equation to solve:

$$[C]\left\{\frac{dT}{dt}\right\} + [K]\{T\} = [F]$$

Heat Transient Equation: PGD

Moving to PGD

$$\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega T^* \frac{\partial^2 T}{\partial x^2} d\Omega = \int_\Gamma T^* f(x,t) d\Omega$$

Heat Transient Equation: PGD

Moving to PGD

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Applying Green's theorem

$$\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega$$

Heat Transient Equation: PGD

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Applying Green's theorem

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Separated representation of the solution:

$$T(x,t) = \sum_{i=1}^{n} [F_i(x) G_i(t)] + R(x) S(t)$$

Heat Transient Equation: PGD

Linear function approximations:

$$F_i(x) = N_1 F_i + N_2 F_i$$

Heat Transient Equation: PGD

Linear function approximations:

$$F_i(x) = N_1 F_i + N_2 F_i$$

Element matrix interpolation:

$$T(x,t) = \sum_{i=1}^{n} [N^T F_i \, M^T G_i] + N^T R \, M^T S$$

Heat Transient Equation: PGD

Linear function approximations:

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$$T(x,t) = \sum_{i=1}^{n} [N^T F_i \, M^T G_i] + N^T R \, M^T S$$

Separated representation also applied to T*:

$$T^*(x,t) = R^*(x) S^*(t) = R^*(x) S(t) + R(x) S^*(t)$$

Heat Transient Equation: PGD

Linear function approximations:

$$F_i(x) = N_1 F_i + N_2 F_i$$

Element matrix interpolation:

$$T(x,t) = \sum_{i=1}^{n} [N^T F_i \, M^T G_i] + N^T R \, M^T S$$

Separated representation also applied to T*:

$$T^*(x,t) = R^*(x) S^*(t) = R^*(x) S(t) + R(x) S^*(t)$$

Same matrix interpolation:

$$T^*(x,t) = N^T R^* \, M^T S + N^T R \, M^T S^*$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Polynomial interpolation:

$$\frac{\partial T}{\partial t} = \sum_{i=1}^{n} [N^T F_i \, dM^T G_i] + [N^T R \, dM^T S]$$

$$\frac{\partial T}{\partial x} = \sum_{i=1}^{n} [dN^T F_i \, M^T G_i] + [dN^T R \, M^T S]$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Polynomial interpolation:

$$\frac{\partial T^*}{\partial t} = \sum_{i=1}^{n} [N^T R^* \, dM^T S] + [N^T R \, dM^T S^*]$$

$$\frac{\partial T^*}{\partial x} = \sum_{i=1}^{n} [dN^T R^* \, M^T S] + [dN^T R \, M^T S^*]$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Polynomial interpolation also to the source term:

$$T^* f(x,t) = [N^T R^* \, M^T S + N^T R \, M^T S^*] \sum_{h=1}^{m} [N^T a_h M^T b_h]$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x}\frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Grouping all terms:

$$\int_\Omega \Bigg[[N^T R^* \, M^T S + N^T R \, M^T S^*] \rho C_p \Big[\sum_{i=1}^{n} [N^T F_i \, dM^T G_i] + [N^T R \, dM^T S] \Big] \Bigg] d\Omega +$$

$$\int_\Omega \Bigg[[dN^T R^* \, M^T S + dN^T R \, M^T S^*] K_x \Big[\sum_{i=1}^{n} [dN^T F_i \, M^T G_i] + [dN^T R \, M^T S] \Big] \Bigg] d\Omega =$$

$$\int_\Gamma \Bigg[[N^T R^* \, M^T S + N^T R \, M^T S^*] \Big[\sum_{h=1}^{m} [N^T a_h \, M^T b_h] \Big] \Bigg] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Alternating Direction algorithm, assuming S known, looking for R (it implies S* -> 0):

$$\int_\Omega \left[[N^T R^* \, M^T S + N^T R \, M^T S^*] \rho C_p \Big[\sum_{i=1}^{n} [N^T F_i \, dM^T G_i] + [N^T R \, dM^T S] \Big] \right] d\Omega +$$

$$\int_\Omega \left[[dN^T R^* \, M^T S + dN^T R \, M^T S^*] K_x \Big[\sum_{i=1}^{n} [dN^T F_i \, M^T G_i] + [dN^T R \, M^T S] \Big] \right] d\Omega =$$

$$\int_\Gamma \left[[N^T R^* \, M^T S + N^T R \, M^T S^*] \Big[\sum_{h=1}^{m} [N^T a_h \, M^T b_h] \Big] \right] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Alternating Direction algorithm, assuming S known, looking for R (it implies S* -> 0):

$$\int_\Omega \left[[N^T R^* \, M^T S + \cancel{N^T R \, M^T S^*}] \rho C_p \Big[\sum_{i=1}^{n} [N^T F_i \, dM^T G_i] + [N^T R \, dM^T S] \Big] \right] d\Omega +$$

$$\int_\Omega \left[[dN^T R^* \, M^T S + \cancel{dN^T R \, M^T S^*}] K_x \Big[\sum_{i=1}^{n} [dN^T F_i \, M^T G_i] + [dN^T R \, M^T S] \Big] \right] d\Omega =$$

$$\int_\Gamma \left[[N^T R^* \, M^T S + \cancel{N^T R \, M^T S^*}] \Big[\sum_{h=1}^{m} [N^T a_h \, M^T b_h] \Big] \right] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Estimating R:

$$\int_\Omega \Big[[N^T R^* \, M^T S] \rho C_p \sum_{i=1}^{n} [N^T F_i \, d M^T G_i] \Big] d\Omega + \int_\Omega \Big[[N^T R^* \, M^T S] \rho C_p [N^T R \, d M^T S] \Big] d\Omega +$$

$$\int_\Omega \Big[[d N^T R^* \, M^T S] K_x \sum_{i=1}^{n} [d N^T F_i \, M^T G_i] \Big] d\Omega + \int_\Omega \Big[[d N^T R^* \, M^T S] K_x [d N^T R \, M^T S] \Big] d\Omega =$$

$$\int_\Gamma \left[[N^T R^* \, M^T S] \Big[\sum_{h=1}^{m} [N^T a_h \, M^T b_h] \Big] \right] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x}\frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Estimating R:

$$\rho Cp \sum_{i=1}^{n} \Big[\int_\Omega [{R^*}^T N\, N^T F_i][S^T M\, dM^T G_i] \Big] + \rho Cp \int_\Omega \Big[[{R^*}^T N\, N^T R][S^T M\, dM^T S] \Big] +$$

$$K_x \sum_{i=1}^{n} \Big[\int_\Omega [{R^*}^T dN\, dN^T F_i][S^T M\, M^T G_i] \Big] + K_x \int_\Omega \Big[[{R^*}^T dN\, dN^T R][S^T M\, M^T S] \Big] =$$

GROUPING!

$$\sum_{h=1}^{m} \Big[\int_\Gamma [{R^*}^T N\, N^T a_h][S^T M\, M^T b_h] \Big] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Estimating R:

$$\rho Cp \sum_{i=1}^{n} \left[\int_\Omega [{R^*}^T N\, N^T F_i][S^T M\, M^T G_i] \right] + \rho Cp \int_\Omega \left[[{R^*}^T N\, N^T R][S^T M\, dM^T S] \right] +$$

$$K_x \sum_{i=1}^{n} \left[\int_\Omega [{R^*}^T dN\, dN^T F_i][S^T M\, M^T G_i] \right] + K_x \int_\Omega \left[[{R^*}^T dN\, dN^T R][S^T M\, M^T S] \right] =$$

$$\sum_{h=1}^{m} \left[\int_\Gamma [{R^*}^T N\, N^T a_h][S^T M\, M^T b_h] \right] d\Gamma$$

SEPARATE VARIABLES!!!

GROUPING!

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Estimating R:

$$\rho Cp \sum_{i=1}^{n} \Big[\int_\Omega [\cancel{R^{*T}} N\, N^T F_i][S^T M\, dM^T G_i] \Big] + \rho Cp \int_\Omega \Big[[\cancel{R^{*T}} N\, N^T R][S^T M\, dM^T S] \Big] +$$

$$K_x \sum_{i=1}^{n} \Big[\int_\Omega [\cancel{R^{*T}} dN\, dN^T F_i][S^T M\, M^T G_i] \Big] + K_x \int_\Omega \Big[[\cancel{R^{*T}} dN\, dN^T R][S^T M\, M^T S] \Big] =$$

$$\sum_{h=1}^{m} \Big[\int_\Gamma [\cancel{R^{*T}} N\, N^T a_h][S^T M\, M^T b_h] \Big] d\Gamma$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x}\frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Estimating R:

$$\rho Cp \sum_{i=1}^{n} \Big[\int_\Omega [N\, N^T F_i][S^T M\, dM^T G_i] \Big] + \rho Cp \int_\Omega \Big[[N\, N^T R][S^T M\, dM^T S] \Big] +$$

$$K_x \sum_{i=1}^{n} \Big[\int_\Omega [dN\, dN^T F_i][S^T M\, M^T G_i] \Big] + K_x \int_\Omega \Big[[dN\, dN^T R][S^T M\, M^T S] \Big] -$$

$$- \sum_{h=1}^{m} \Big[\int_\Gamma [N\, N^T a_h][S^T M\, M^T b_h] \Big] d\Gamma = 0$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

Matrix definition (for clarity):

$$C_M = \int_{\Omega_t} M\, dM^T dt \qquad M_N = \int_{\Omega_x} N\, N^T dx$$

$$K_N = \int_{\Omega_x} dN\, dN^T dx \qquad M_M = \int_{\Omega_t} M\, M^T dt$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x}\frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

$$\boxed{\begin{aligned} R = \Big[\rho Cp[S^T C_M S M_N] + K_x[S^T M_M S K_N]\Big]^{-1} \Bigg[\sum_{h=1}^{m}[M_N a_h S^T M_M b_h] - \\ - \sum_{i=1}^{n}\Big[\rho Cp[M_N F_i S^T C_M G_i] + K_x[K_N F_i S^T M_M G_i]\Big]\Bigg] \end{aligned}}$$

Heat Transient Equation: PGD

$$\boxed{\rho c_p \int_\Omega T^* \frac{\partial T}{\partial t} d\Omega + k_x \int_\Omega \frac{\partial T^*}{\partial x}\frac{\partial T}{\partial x} d\Omega = \int_\Gamma T^* f(x,t) d\Omega}$$

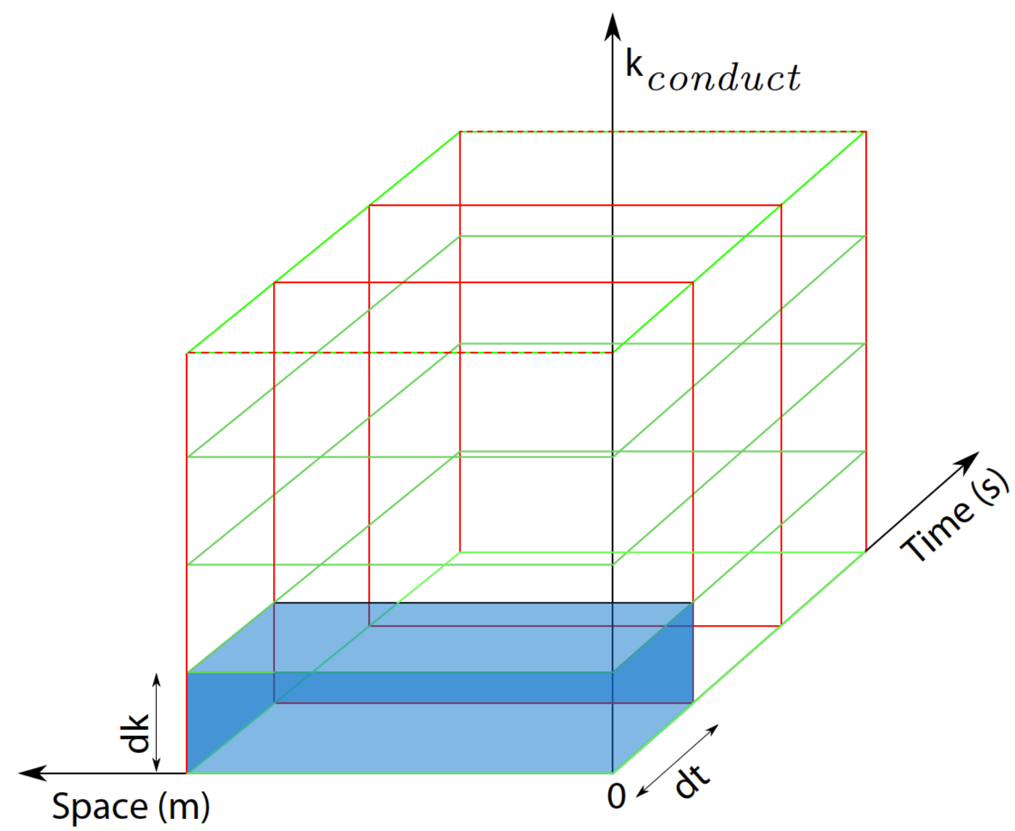
$$\boxed{\begin{aligned} S = \Big[\rho Cp[R^T M_N R C_M] + K_x[R^T K_N R M_M]\Big]^{-1} \Bigg[\sum_{h=1}^{m} [M_M b_h R^T M_N a_h] - \\ - \sum_{i=1}^{n} \Big[\rho Cp[C_M G_i R^T M_N F_i] + K_x[M_M G_i R^T K_N F_i]\Big]\Bigg] \end{aligned}}$$

Heat Transient Equation: PGD

Algorithm 2.1: Pseudo-code of the greedy algorithm used by the PGD method.

Initialization;
While (NMode $<$ maxNModes) & (Tol $>$ Error) **do**
{
  EnrichmentInitialization;
  While (Iters $<$ maxIters) & (enrTol $>$ enrError) **do**
  {
    $R \leftarrow$ EnrichFromS;
    $S \leftarrow$ EnrichFromR;
    enrError $\leftarrow$ enrErrorEstimation;
    Iters++;
  }
  F(NMode) $\leftarrow R$;
  G(NMode) $\leftarrow S$;
  Error $\leftarrow$ ErrorEstimation;
  NMode++;
}

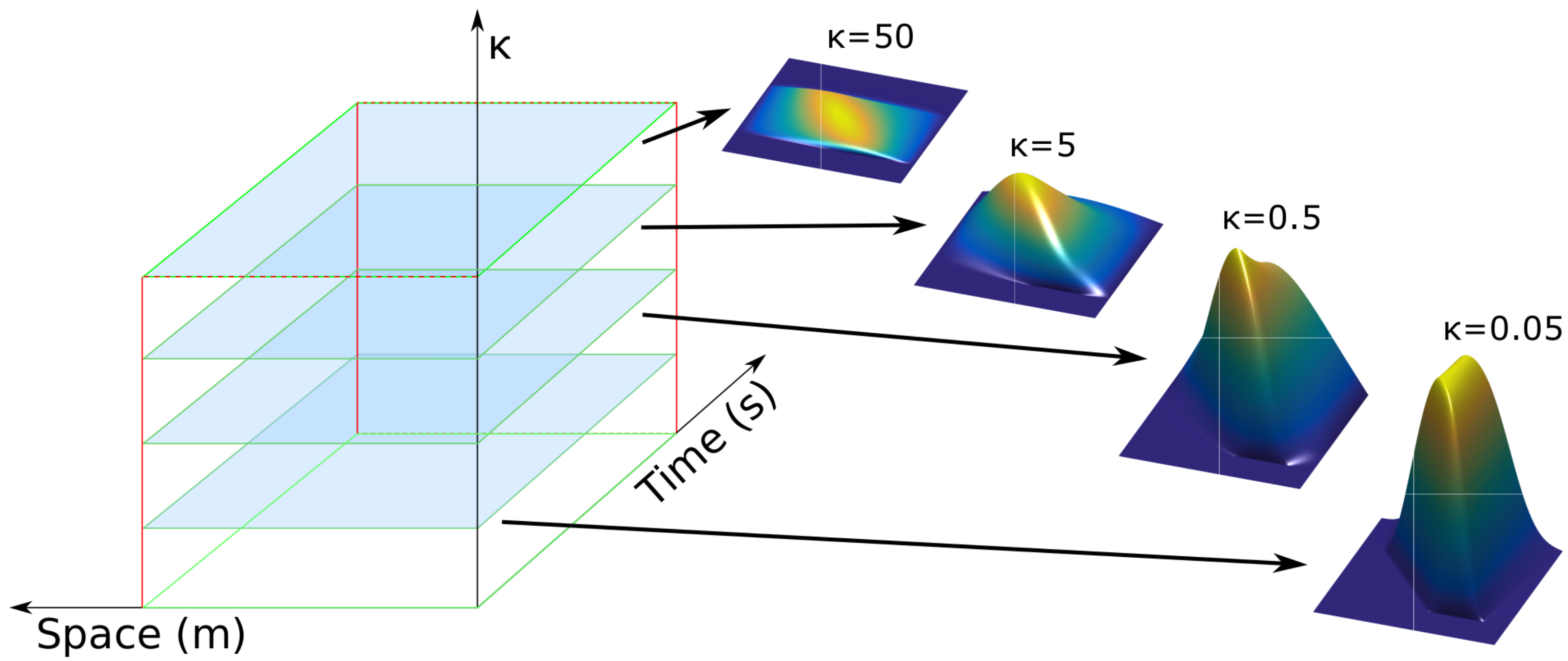
Parametric Heat Transient Equation



Parametric Heat Transient Equation

$$\boxed{\rho c_p \frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t) = 0}$$

$$u(x,t,k) \approx \sum_{i=1}^{p} [F_i(x) \cdot G_i(t) \cdot H_i(k)]$$



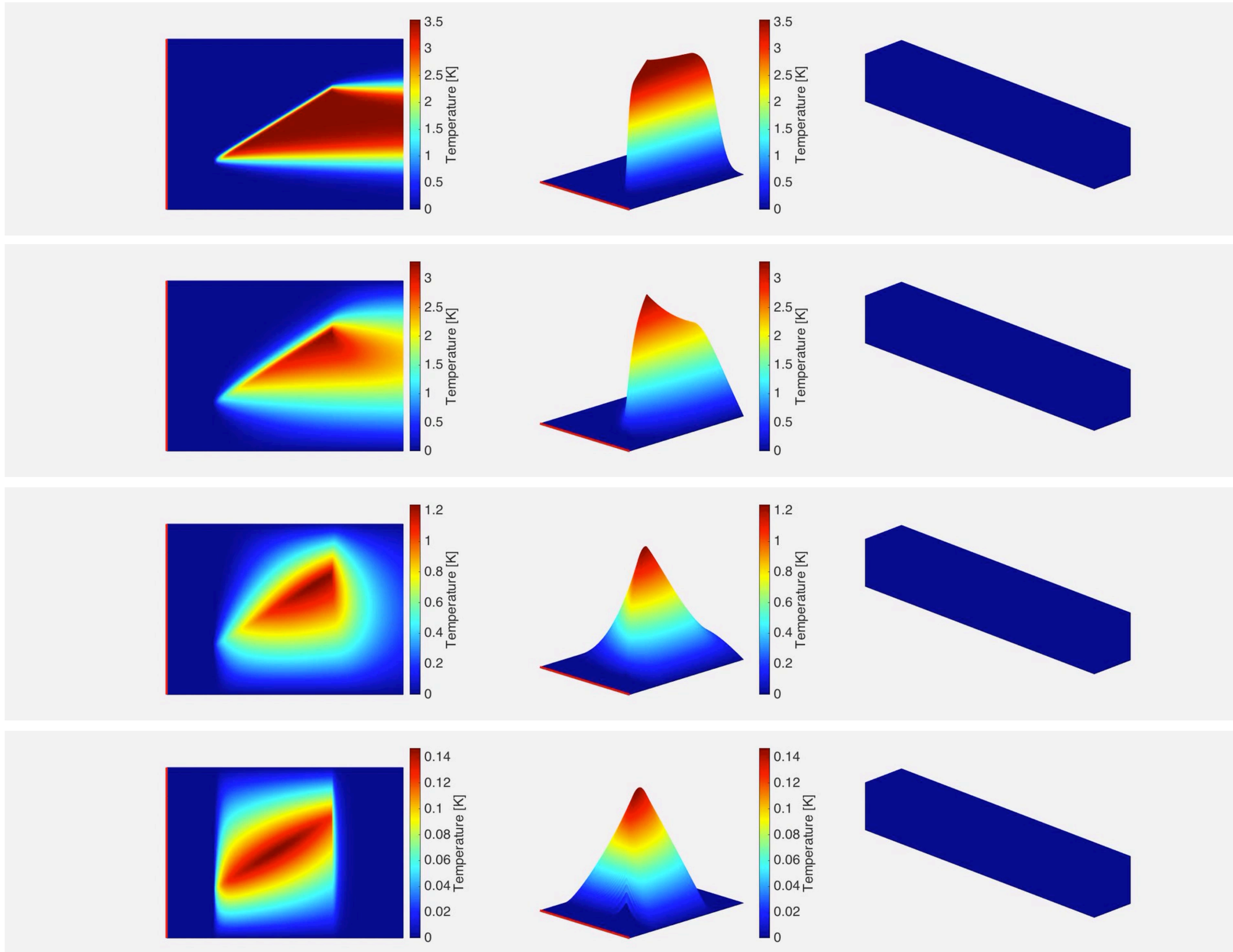
Parametric Heat Transient Equation

$$k = 0.05 \, \frac{W}{m \cdot K}$$

$$k = 0.5 \, \frac{W}{m \cdot K}$$

$$k = 5 \, \frac{W}{m \cdot K}$$

$$k = 50 \, \frac{W}{m \cdot K}$$



Lecture 2

Introduction to Proper Generalized Decomposition Techniques

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UKACM 2021 Conference

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