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Bridging methodologies for the computational simulation of fluids and solids

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4^{th} **ACME School: "Low or high-order: this is the question!"**

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Outline

1. Introduction
   - Motivation
   - Traditional dynamic formulation
   - Aims
2. Governing equation
   - Conservation laws for solid dynamics
   - System of conservation laws
   - Available numerical methodologies
3. Numerical methods
   - Jameson-Schmidt-Turkel (JST) scheme
   - Petrov-Galerkin (PG) spatial discretisation
   - Temporal discretisation
4. Numerical results
5. Extension to FSI
   - Immersed Structural Potential Method (ISPM)
   - Key ingredients
6. Conclusions

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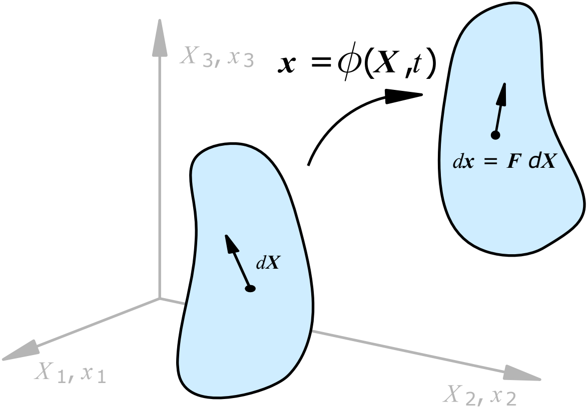
Motion of a discretised solid



$$\boldsymbol{x} = \phi(\boldsymbol{X}, t); \qquad \boldsymbol{v} = \dot{\boldsymbol{x}}; \qquad \boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x}$$

Traditional dynamic formulation

- Classical dynamic equilibrium equation:

$$\rho_0 \dot{\boldsymbol{v}} - DIV \boldsymbol{P}(\boldsymbol{\nabla}_0 \boldsymbol{x}) = \rho_0 \boldsymbol{b}$$
$$\boldsymbol{v} = \dot{\boldsymbol{x}}$$

- And constitutive model:

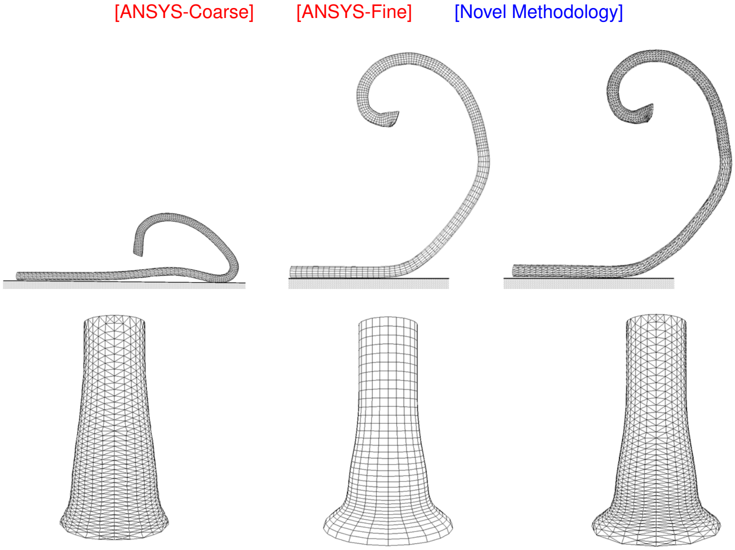
$$\boldsymbol{P} = \frac{\partial \psi(\boldsymbol{\nabla}_0 \boldsymbol{x}, \ldots)}{\partial \boldsymbol{F}}$$

- Satisfaction of **compatibility conditions**
- Variational formulation using **linear** finite element interpolation:

$$\sum_b \mathcal{M}_{ab} \dot{\boldsymbol{v}}_b = \int_{\partial V_0} N_a \boldsymbol{t}^B \, dA + \int_{V_0} N_a \rho_0 \boldsymbol{b} \, dV - \int_{V_0} \boldsymbol{P}(\boldsymbol{\nabla}_0 \boldsymbol{x}) \boldsymbol{\nabla}_0 N_a \, dV$$

- **Disadvantages**:
  - Bending difficulty [Bonet, 1998]
  - Volumetric locking [Bonet, 2001]
  - Pressure instability [Gee, 2009]
  - Reduced order of accuracy for strains and stresses

Explicit FE: Locking examples

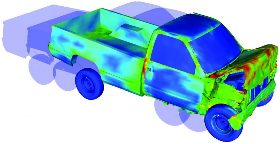


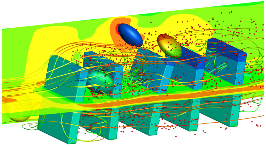
P1/Q0 Tet. P1/Q0 Hex. P1/Q1 Tet.

Aims

Solid community

- Computational Solid Dynamics is a well established subject and there is extensive software available
- Standard displacement-based FE based on
  - Explicit codes
  - Under-integrated hexahedral elements
  - Nodal linear tetrahedral technology
  - Updated Lagrangian formulations
- Shortcomings
  - Overly **stiff** in bending
  - **Locking** difficulties
  - **First** order precision for strains and stresses
  - **Inefficient** in the vicinity of shocks





CFD community

- Many robust techniques are available for linear triangle and tetrahedra
- Mature adaptive mesh generation
- Equal rate of convergence for velocity and pressure
- Robust shock capturing

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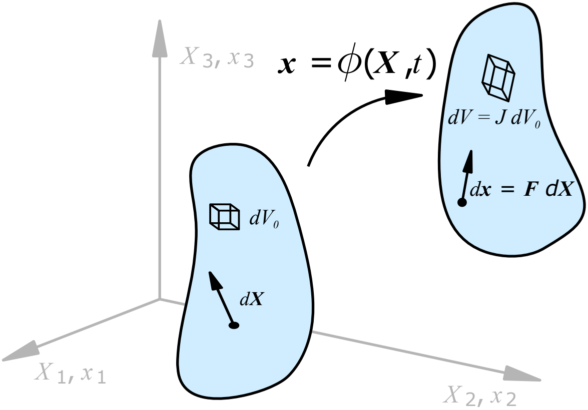
Large strain kinematics: $\boldsymbol{F}$ and J



Fibre map $\boldsymbol{F}$ Volume map J

Conservation laws

- **Conservation of linear momentum**:

$$\frac{d}{dt}\int_{V_0} \boldsymbol{p}\, dV_0 = \int_{V_0} \rho_0 \boldsymbol{b}\, dV_0 + \int_{\partial V_0} \boldsymbol{t}\, dA; \qquad \boldsymbol{t} = \boldsymbol{P}\boldsymbol{N}$$

- In differential form:

$$\frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \boldsymbol{P} = \rho_0 \boldsymbol{b}$$

- **Conservation of deformation gradient**:

$$\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{\phi} \implies \int_{V_0} \boldsymbol{F}\, dV_0 = \int_{\partial V_0} \boldsymbol{\phi} \otimes d\boldsymbol{A} \implies \frac{d}{dt}\int_{V_0} \boldsymbol{F}\, dV_0 = \int_{\partial V_0} \frac{1}{\rho_0}\boldsymbol{p} \otimes d\boldsymbol{A}$$

- In differential form:

$$\frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \left(\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{I}\right) = \boldsymbol{0}$$

- **However, $\boldsymbol{p}$-$\boldsymbol{F}$ formulation suffers numerical instabilities in nearly incompressible deformations** [Gil, Lee, Bonet and Aguirre, 2014]

Conservation laws for solid dynamics

- **Conservation of Jacobian**:

$$\frac{d}{dt}\int_V dV = \int_{\partial V} \boldsymbol{v}\cdot d\boldsymbol{a} \Longrightarrow \frac{d}{dt}\int_{V_0} J\, dV_0 = \int_{\partial V_0} \left(\boldsymbol{H}_F^T \boldsymbol{v}\right)\cdot d\boldsymbol{A}; \qquad \boldsymbol{H}_F = (\det \boldsymbol{F})\boldsymbol{F}^{-T}$$

- In differential form:

$$\frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \boldsymbol{P} = \rho_0 \boldsymbol{b}$$
$$\frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \left(\frac{1}{\rho_0}\boldsymbol{p}\otimes \boldsymbol{I}\right) = \boldsymbol{0}$$
$$\frac{\partial J}{\partial t} - \boldsymbol{\nabla}_0 \cdot \left(\frac{1}{\rho_0}\boldsymbol{H}_F^T \boldsymbol{p}\right) = 0$$

- With **Curl Free** conditions:

$$\boldsymbol{\nabla}_0 \times \boldsymbol{F} = \boldsymbol{0}$$

- And constitutive model:

$$\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F}, J)$$

- Appropriate initial and boundary conditions

System of conservation laws

- Using the combined notation:

$$\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ J \end{bmatrix} + \boldsymbol{\nabla}_0 \cdot \begin{bmatrix} -\boldsymbol{P}(\boldsymbol{F}, J) \\ -\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{I} \\ -\frac{1}{\rho_0}\boldsymbol{H}_{\boldsymbol{F}}^T \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \rho_0 \boldsymbol{b} \\ \boldsymbol{0} \\ 0 \end{bmatrix}$$

- **Energy equation** is added if necessary:

$$\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ J \\ E \end{bmatrix} + \boldsymbol{\nabla}_0 \cdot \begin{bmatrix} -\boldsymbol{P}(\boldsymbol{F}, J, \dots) \\ -\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{I} \\ -\frac{1}{\rho_0}\boldsymbol{H}_{\boldsymbol{F}}^T \boldsymbol{p} \\ \boldsymbol{Q} - \frac{1}{\rho_0}\boldsymbol{P}^T \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \rho_0 \boldsymbol{b} \\ \boldsymbol{0} \\ 0 \\ s \end{bmatrix}$$

- Or in standard form:

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \boldsymbol{\nabla}_0 \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}) = \boldsymbol{\mathcal{S}}; \quad \boldsymbol{\mathcal{U}} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ J \\ E \end{bmatrix}; \; \boldsymbol{\mathcal{F}} = \begin{bmatrix} -\boldsymbol{P}(\boldsymbol{F}, J, \dots) \\ -\frac{1}{\rho_0}\boldsymbol{p} \otimes \boldsymbol{I} \\ -\frac{1}{\rho_0}\boldsymbol{H}_{\boldsymbol{F}}^T \boldsymbol{p} \\ \boldsymbol{Q} - \frac{1}{\rho_0}\boldsymbol{P}^T \boldsymbol{p} \end{bmatrix}; \; \boldsymbol{\mathcal{S}} = \begin{bmatrix} \rho_0 \boldsymbol{b} \\ \boldsymbol{0} \\ 0 \\ s \end{bmatrix}$$

Tailor-made CFD methodologies have been developed

Adapted CFD technologies for solid dynamics

- Following are the discretisation techniques available for solid dynamics:

(SU) **Two-Step Taylor-Galerkin FEM** [Karim, Lee, Gil and Bonet, 2011]

(SU) **Upwind Cell Centred FVM** [Lee, Gil and Bonet, 2012]

(MIT) **Hybridizable Discontinuous Galerkin FEM** [Nguyen and Peraire, 2012]

(LJLL) **Godunov Lagrangian FVM** [Kluth and Després, 2012]

(SU) **JST Vertex Centred FVM** [Aguirre, Gil, Bonet and Carreño, 2013]

(SU) **Petrov-Galerkin FEM** [Lee, Gil and Bonet, 2013]

(SU) **Fractional step Petrov-Galerkin FEM** [Gil, Lee, Bonet and Aguirre, 2014]

(SU) **Polyconvex Petrov-Galerkin FEM** [Bonet, Gil, Lee, Aguirre and Ortigosa, 2015]

(SU) **Upwind Vertex Centred FVM** [Aguirre, Gil, Bonet and Lee, 2015]

(SU) **Polyconvex (Fractional) Hydrocode** [Lee, Gil, Bonet and Ortigosa, Under review]

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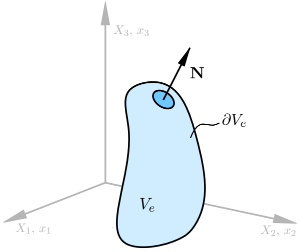
Jameson-Schmidt-Turkel (JST) Algorithm

The Jameson-Schmidt-Turkel (JST) scheme

Large scale simulations → **JST** [Jameson et al, 1981]

- Large success within the CFD community
- **High efficiency**
  - **Edge** based
  - **Vertex centred** (el/nodes ≃ 5 in 3D)
  - No need of **linear reconstruction** and **limiters**
- Suitable for unstructured meshes
- Built-in **shock capturing** term



Standard in **FVM**: integrate within control volume + divergence theorem

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \boldsymbol{\nabla}_0 \cdot \boldsymbol{\mathcal{F}} = \mathbf{0} \longrightarrow \int_{V_e} \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t}\, dV = \int_{V_e} -\boldsymbol{\nabla}_0 \cdot \boldsymbol{\mathcal{F}}\, dV \Rightarrow V_e \frac{d\boldsymbol{\mathcal{U}}_e}{dt} = \int_{\partial V_e} -\boldsymbol{\mathcal{F}}\boldsymbol{N}\, d\Gamma$$

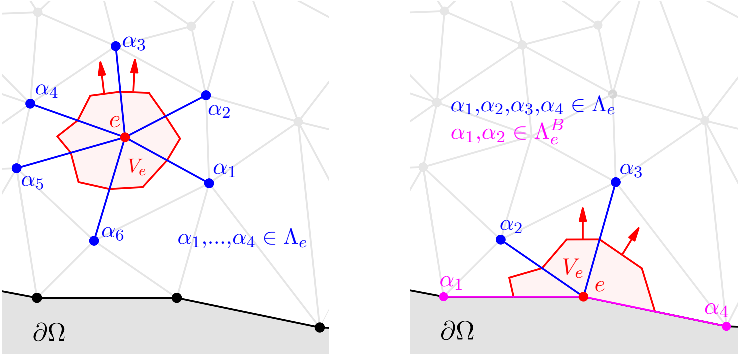
Specific **JST**: Discretise in space using **centred differences** + **artificial dissipation**

$$V_e \frac{d\boldsymbol{\mathcal{U}}_e}{dt} = -\left(\sum_{\alpha \in \Lambda_e} \frac{\boldsymbol{\mathcal{F}}_e + \boldsymbol{\mathcal{F}}_\alpha}{2} \boldsymbol{C}^{e\alpha} + \sum_{\gamma \in \Gamma_e^B} \boldsymbol{\mathcal{F}}_\gamma \boldsymbol{N}^\gamma \frac{A^\gamma}{3} \right) + \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{U}}_e)$$

Dual mesh construction

Medial dual approach

2D: connection of element centroids with edge midpoints



$$\mathbf{C}^{e\alpha} = \sum_{k\in\Gamma_{e\alpha}} A_k \mathbf{N}_k, \qquad \text{edge area vector}$$

Discretised governing equations

Adapted JST scheme to the governing equations

$$\int_{V_e} \frac{\partial}{\partial t} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \\ J \end{bmatrix} dV = \int_{V_e} \boldsymbol{\nabla}_0 \cdot \begin{bmatrix} \boldsymbol{P}(\boldsymbol{F}, J) \\ \boldsymbol{v} \otimes \boldsymbol{I} \\ \boldsymbol{H}_F^T \boldsymbol{v} \end{bmatrix} dV \Rightarrow V_e \frac{d}{dt} \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{F}_e \\ J_e \end{bmatrix} = \int_{\partial V_e} \begin{bmatrix} \boldsymbol{PN} \\ \boldsymbol{v} \otimes \boldsymbol{N} \\ (\boldsymbol{H}_F^T \boldsymbol{v}) \cdot \boldsymbol{N} \end{bmatrix} d\Gamma$$

$$V_e \frac{d\boldsymbol{p}_e}{dt} = \sum_{\alpha \in \Lambda_e} \frac{1}{2} (\boldsymbol{P}_e + \boldsymbol{P}_\alpha) \boldsymbol{C}^{e\alpha} + \sum_{\gamma \in \Gamma_e^B} \hat{\boldsymbol{t}}^\gamma \frac{A^\gamma}{3} + \boldsymbol{\mathcal{D}}(\boldsymbol{p}_e)$$

$$V_e \frac{d\boldsymbol{F}_e}{dt} = \sum_{\alpha \in \Lambda_e} \frac{\boldsymbol{v}_e + \boldsymbol{v}_\alpha}{2} \otimes \boldsymbol{C}^{e\alpha} + \sum_{\gamma \in \Gamma_e^B} (\hat{\boldsymbol{v}}^\gamma \otimes \boldsymbol{N}^\gamma) \frac{A^\gamma}{3}$$

$$V_e \frac{dJ_e}{dt} = \sum_{\alpha \in \Lambda_e} \frac{\boldsymbol{H}_{F,e}^T \boldsymbol{v}_e + \boldsymbol{H}_{F,\alpha}^T \boldsymbol{v}_\alpha}{2} \cdot \boldsymbol{C}^{e\alpha} + \sum_{\gamma \in \Gamma_e^B} (\hat{\boldsymbol{v}}^\gamma \cdot \boldsymbol{H}_F^\gamma \boldsymbol{N}^\gamma) \frac{A^\gamma}{3} + \boldsymbol{\mathcal{D}}(J_e)$$

- Dissipation only added to the **first and third equations**
- Addition of dissipation to the second equation affects **compatibility conditions** $(\boldsymbol{\nabla}_0 \times \boldsymbol{F} \neq \boldsymbol{0})$

$$\boldsymbol{\mathcal{D}}(\boldsymbol{U}_e) = \sum_{\alpha \in \Lambda_e} \varepsilon^{(2)} \Psi_{e\alpha} \theta_{e\alpha} (\boldsymbol{U}_\alpha - \boldsymbol{U}_e) - \sum_{\alpha \in \Lambda_e} \varepsilon^{(4)} \Psi_{e\alpha} \theta_{e\alpha} (\boldsymbol{L}(\boldsymbol{U}_\alpha) - \boldsymbol{L}(\boldsymbol{U}_e))$$

Petrov-Galerkin (PG) Algorithm

Petrov-Galerkin formulation

- Variational statement of **Bubnov-Galerkin** formulation:

$$\int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{R}}\, dV = 0; \qquad \boldsymbol{\mathcal{R}} = \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \boldsymbol{\nabla}_0 \cdot \boldsymbol{\mathcal{F}} - \boldsymbol{\mathcal{S}}; \qquad \delta\boldsymbol{\mathcal{V}} = \begin{bmatrix} \delta\boldsymbol{v} \\ \delta\boldsymbol{P} \\ \delta p \end{bmatrix}$$

- Integration by parts gives (**unstable**) [Morgan and Peraire, 1998]:

$$\int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t}\, dV = \int_{V_0} \boldsymbol{\mathcal{F}} : \boldsymbol{\nabla}_0 \delta\boldsymbol{\mathcal{V}}\, dV - \int_{\partial V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{F}}\boldsymbol{N}\, dA + \int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{S}}\, dV$$

- Define **stabilised PG formulation** [Hughes et al, 1986]:

$$\int_{V_0} \delta\boldsymbol{\mathcal{V}}^{st} \cdot \boldsymbol{\mathcal{R}}\, dV = 0; \qquad \delta\boldsymbol{\mathcal{V}}^{st} = \begin{bmatrix} \delta\boldsymbol{v}^{st} \\ \delta\boldsymbol{P}^{st} \\ \delta q^{st} \end{bmatrix}$$

Perturbation

- Stabilised test function space is generally defined by

$$\delta\boldsymbol{\mathcal{V}}^{st} = \delta\boldsymbol{\mathcal{V}} + \underbrace{\boldsymbol{\tau}^T \left(\frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial \boldsymbol{\mathcal{U}}}\right)^T \frac{\partial \delta\boldsymbol{\mathcal{V}}}{\partial X_I}}_{\textbf{Perturbation}}$$

- Define flux Jacobian matrix:

$$\frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial \boldsymbol{\mathcal{U}}} = \begin{pmatrix} \mathbf{0}_{3\times3} & -\boldsymbol{\mathcal{C}}_I & -\kappa\left[\boldsymbol{H_F}\right]_I \\ -\frac{1}{\rho_0}\boldsymbol{I}_I & \mathbf{0}_{9\times9} & \mathbf{0}_{9\times1} \\ -\frac{1}{\rho_0}\left[\boldsymbol{H_F}\right]_I & -\frac{\partial\left(\boldsymbol{v}\cdot\left[\boldsymbol{H_F}\right]_I\right)}{\partial \boldsymbol{F}} & 0 \end{pmatrix}$$

- Assuming $\boldsymbol{\tau}$ (intrinsic time scale) a diagonal matrix for simplicity:

$$\delta\boldsymbol{\mathcal{V}}^{st} := \begin{bmatrix} \delta\boldsymbol{v}^{st} \\ \delta\boldsymbol{P}^{st} \\ \delta q^{st} \end{bmatrix} = \begin{bmatrix} \delta\boldsymbol{v} - \frac{\tau_{pF}}{\rho_0} DIV\delta\boldsymbol{P} - \frac{\tau_{pJ}}{\rho_0}\boldsymbol{H_F}\boldsymbol{\nabla}_0\delta q \\ \delta\boldsymbol{P} - \tau_{Fp}\boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} - \tau_{FJ}\left(\boldsymbol{v}\otimes\boldsymbol{\nabla}_0\delta q\right) : \frac{\partial \boldsymbol{H_F}}{\partial \boldsymbol{F}} \\ \delta q - \tau_{Jp}\kappa\boldsymbol{H_F} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \end{bmatrix}$$

- Bubnov-Galerkin is recovered by setting stabilisation $\boldsymbol{\tau} = \mathbf{0}$

Petrov-Galerkin stabilisation

- Weak statement of **stabilised Petrov-Galerkin (PG) formulation**:

$$0 = \int_{V_0} \left[\delta\boldsymbol{\mathcal{V}} + \left(\frac{\partial \boldsymbol{\mathcal{F}}}{\partial \boldsymbol{\mathcal{U}}} \boldsymbol{\tau} \right)^T \boldsymbol{\nabla}_0 \delta\boldsymbol{\mathcal{V}} \right] \cdot \boldsymbol{\mathcal{R}} \, dV$$

$$= \underbrace{\int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{R}} \, dV}_{\textbf{Bubnov-Galerkin}} + \underbrace{\int_{V_0} \left[\frac{\partial \boldsymbol{\mathcal{F}}}{\partial \boldsymbol{\mathcal{U}}} \boldsymbol{\tau} \boldsymbol{\mathcal{R}} \right] : \boldsymbol{\nabla}_0 \delta\boldsymbol{\mathcal{V}} \, dV}_{\textbf{Petrov-Galerkin stabilisation}}$$

- Integration by parts gives:

$$\int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} \, dV = \int_{V_0} \underbrace{\left[\boldsymbol{\mathcal{F}} - \frac{\partial \boldsymbol{\mathcal{F}}}{\partial \boldsymbol{\mathcal{U}}} \boldsymbol{\tau} \boldsymbol{\mathcal{R}} \right]}_{\boldsymbol{\mathcal{F}}^{st}} : \boldsymbol{\nabla}_0 \delta\boldsymbol{\mathcal{V}} \, dV - \int_{\partial V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{F}} \boldsymbol{N} \, dA + \int_{V_0} \delta\boldsymbol{\mathcal{V}} \cdot \boldsymbol{\mathcal{S}} \, dV$$

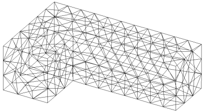
- The stabilised flux $\boldsymbol{\mathcal{F}}^{st}$ can be more generally defined as (in equivalent to **Variational Multi-Scale (VMS)** [Scovazzi, 2012]):

$$\boldsymbol{\mathcal{F}}^{st} = \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}^{st}); \qquad \boldsymbol{\mathcal{U}}^{st} = \boldsymbol{\mathcal{U}} + \boldsymbol{\mathcal{U}}'; \qquad \boldsymbol{\mathcal{U}}' = -\boldsymbol{\tau}\boldsymbol{\mathcal{R}}$$

Finite element discretisation

Linear interpolation for primary variables:

$$\boldsymbol{v} = \sum_a \boldsymbol{v}_a N_a; \qquad \boldsymbol{F} = \sum_a \boldsymbol{F}_a N_a; \qquad J = \sum_a J_a N_a$$



$$\sum_b \mathcal{M}_{ab}\dot{\boldsymbol{p}}_b = \int_{\partial V_0} N_a \boldsymbol{t}^B \, dA + \int_{V_0} N_a \rho_0 \boldsymbol{b} \, dV - \int_{V_0} \boldsymbol{P}(\boldsymbol{F}^{st}, J^{st}) \boldsymbol{\nabla}_0 N_a \, dV$$

$$\sum_b \mathcal{M}_{ab}\dot{\boldsymbol{F}}_b = \int_{\partial V_0} N_a (\boldsymbol{v}^B \otimes \boldsymbol{N}) \, dA - \int_{V_0} \boldsymbol{v} \otimes \boldsymbol{\nabla}_0 N_a \, dV$$

$$\sum_b \mathcal{M}_{ab}\dot{J}_b = \int_{\partial V_0} N_a (\boldsymbol{v}^B \cdot \boldsymbol{H}_{\boldsymbol{F}} \boldsymbol{N}) \, dA - \int_{V_0} (\boldsymbol{H}_{\boldsymbol{F}}^T \boldsymbol{v}^{st}) \cdot \boldsymbol{\nabla}_0 N_a \, dV$$

- By construction:

$$\boldsymbol{F}^{st} = \boldsymbol{F} + \tau_{\boldsymbol{F}} \left(\boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{\boldsymbol{F}} \right)$$

$$J^{st} = J + \tau_J \left(\boldsymbol{H} : \boldsymbol{\nabla}_0 \boldsymbol{v} - \dot{J} \right)$$

$$\boldsymbol{v}^{st} = \boldsymbol{v} + \frac{\tau_{\boldsymbol{p}}}{\rho_0} \left(\boldsymbol{\nabla}_0 \cdot \boldsymbol{P} + \rho_0 \boldsymbol{b} - \dot{\boldsymbol{p}} \right)$$

- **Reduce implictness** of the formulation ($\tau_J = \xi_{\boldsymbol{F}} = 0$)

Time Integration

- Integration in time is achieved by means of an explicit two-stage **Total Variation Diminishing** Runge-Kutta (TVD-RK) time integrator:

$$\mathcal{U}_{n+1}^{(1)} = \mathcal{U}_n + \Delta t \dot{\mathcal{U}}_n$$
$$\mathcal{U}_{n+2}^{(2)} = \mathcal{U}_{n+1}^{(1)} + \Delta t \dot{\mathcal{U}}_{n+1}^{(1)}$$
$$\mathcal{U}_{n+1} = \frac{1}{2}\left(\mathcal{U}_n + \mathcal{U}_{n+2}^{(2)}\right)$$

  With a stability constraint

$$\Delta t = \alpha_{CFL} \frac{h_{min}}{c_{max}^n}; \qquad c_{max}^n = \max_a \left(c_{p,a}^n\right)$$

- **Angular momentum** conserving algorithm is required [Lee et al, 2013]
- Geometry increment:

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \frac{\Delta t}{2}\left(\boldsymbol{v}^n + \boldsymbol{v}^{n+1}\right)$$

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Problem statement

Problem description: Unit side cube, linear elasticity, $\rho_0 = 1.1\text{Mg/m}^3$, $E = 1.7 \times 10^7\text{Pa}$, $\nu = 0.45$, $\alpha_{CFL} = 0.3$, $\tau_F = \Delta t$, $\alpha = 0.1$, $\tau_p = 0.2\Delta t$, lumped mass contribution

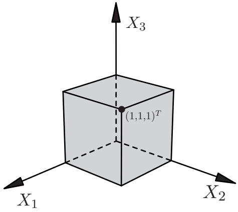
- Given **analytical solution** of the form

$$\boldsymbol{u} = U_0 \cos\left(\frac{\sqrt{3}}{2} c_d \pi t\right) \begin{bmatrix} A \sin\left(\frac{\pi X_1}{2}\right) \cos\left(\frac{\pi X_2}{2}\right) \cos\left(\frac{\pi X_3}{2}\right) \\ B \cos\left(\frac{\pi X_1}{2}\right) \sin\left(\frac{\pi X_2}{2}\right) \cos\left(\frac{\pi X_3}{2}\right) \\ C \cos\left(\frac{\pi X_1}{2}\right) \cos\left(\frac{\pi X_2}{2}\right) \sin\left(\frac{\pi X_3}{2}\right) \end{bmatrix}$$

- Impose IC at t=0, $\boldsymbol{F}^0(\boldsymbol{X}) = \boldsymbol{F}(\boldsymbol{X}, t = 0)$
- Symmetric BC at $X_1 = 0$, $X_2 = 0$ and $X_3 = 0$
- Skew symmetric BC at $X_1 = 1$, $X_2 = 1$ and $X_3 = 1$
- Problem parameters: $A = 2$, $B = -1$, $C = -1$, $U_0 = 5 \times 10^{-4}$, $c_d = \sqrt{\frac{\mu}{\rho_0}}$

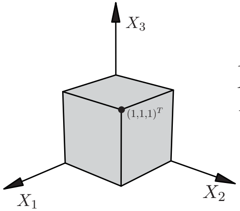
[Swinging Cube] *Solution plotted with displacements scaled* 200 *times*

Symmetric BC



$X_1 = 0$
$X_2 = 0$
$X_3 = 0$

Skew symmetric BC

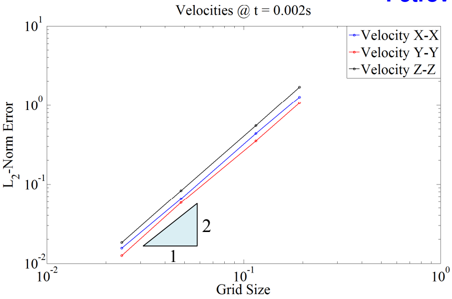


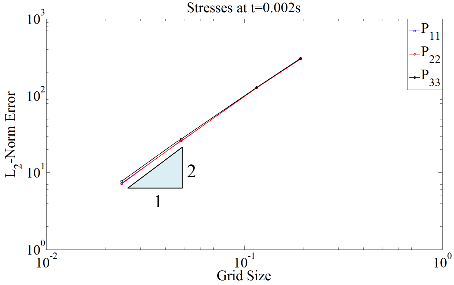
$X_1 = 1$
$X_2 = 1$
$X_3 = 1$

Mesh convergence

- **Problem description**: Unit side cube, linear elasticity, $\rho_0 = 1.1\text{Mg/m}^3$, $E = 1.7 \times 10^7\text{Pa}$, $\nu = 0.45$, $\alpha_{CFL} = 0.3$, $\tau_F = \Delta t$, $\alpha = 0.1$, $\tau_p = 0.2\Delta t$, $A = B = 1$, $C = -2$, $U_0 = 5 \times 10^{-4}$, lumped mass contribution

Petrov Galerkin

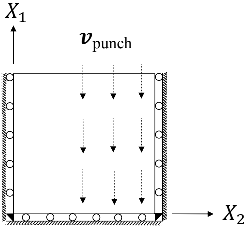




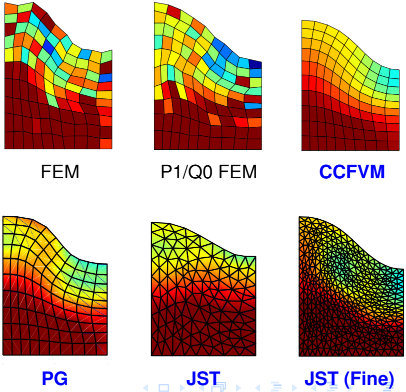
- ✓ L^2 norm convergence analysis
- ✓ Demonstrates the expected accuracy of PG formulation using lumped mass matrix

Spurious mechanisms analysis

Problem description: Unit square plate. Hyperelastic Neo-Hookean material with $\rho_0 = 1.1\text{Mg/m}^3$, $E = 0.017\text{GPa}$, $\nu = 0.45$, $\alpha_{CFL} = 0.5$, $v_{\text{punch}} = 100\text{m/s}$, $\kappa/\mu = 9.1667$, $\Delta t = 1 \times 10^{-4}$ s (and $\Delta t = 2.5 \times 10^{-5}$ s for the fine mesh)



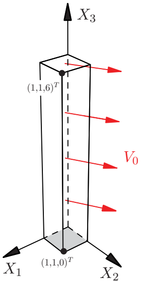
Solution at $t = 0.03$ s

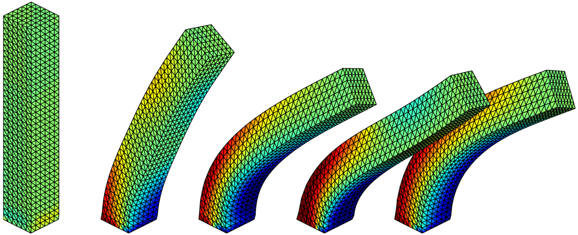


- Absence of **volumetric locking**
- **Smooth** distribution of pressures
- More dissipative than CCFVM

Bending difficulty

Problem description: Column $1 \times 1 \times 6$, $\rho_0 = 1.1$ Mg/m^3, $E = 0.017$ GPa, $\nu = 0.45$, NH model, $\alpha_{CFL} = 0.4$, $\kappa^{(4)} = 1/128$, $V_0 = 10$ m/s with $\theta_{XY} = 30^o$.





JST algoritm

- **Bending dominated problem**
- **Mesh refinement**

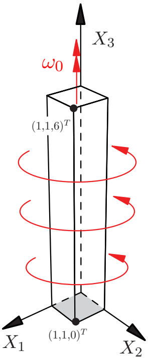
$h = 1/3$ m
$h = 1/6$ m
$h = 1/12$ m

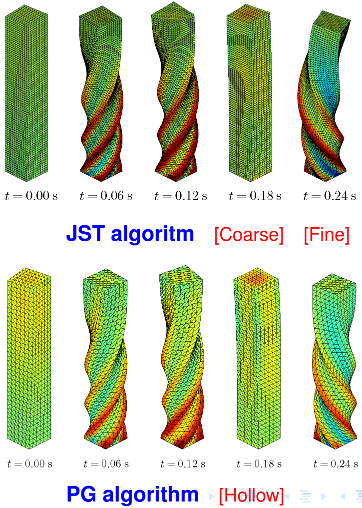
Movies

JST	[Coarse]	[Fine]
PG	[Slender]	[Cylinder]

Robustness

Problem description: Column $1 \times 1 \times 6$, $\rho_0 = 1.1$ Mg/m^3, $E = 0.017$ GPa, $\nu = 0.45$, NH model, $\alpha_{CFL} = 0.4$, $\kappa^{(4)} = 1/128$.

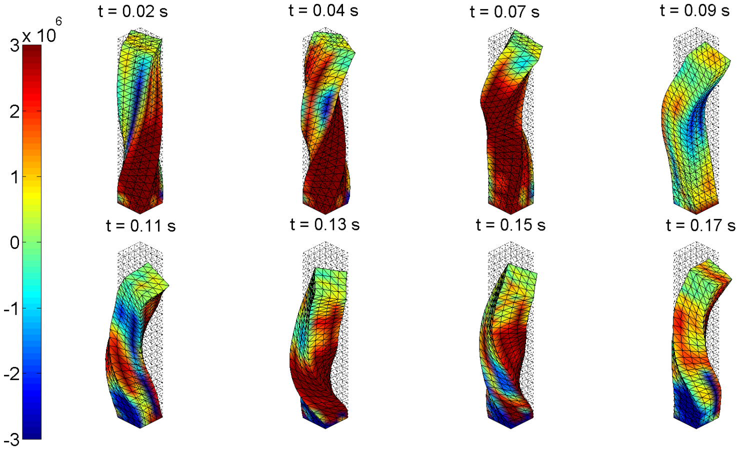




- **Highly nonlinear problems**
- **Robustness of the methodologies**

Limitation of p-F formulation

Volumetric locking and pressure instabilities can be clearly observed



[Twisting Column-0.495]

Outline

1 Introduction
- Motivation
- Traditional dynamic formulation
- Aims

2 Governing equation
- Conservation laws for solid dynamics
- System of conservation laws
- Available numerical methodologies

3 Numerical methods
- Jameson-Schmidt-Turkel (JST) scheme
- Petrov-Galerkin (PG) spatial discretisation
- Temporal discretisation

4 Numerical results

5 Extension to FSI
- Immersed Structural Potential Method (ISPM)
- Key ingredients

6 Conclusions

Extension to FSI

- Incompressible Navier-Stokes equations in the presence of a **solid** structure in Eulerian format:

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}'(\boldsymbol{F}, \boldsymbol{\nabla} \boldsymbol{v}) = \rho \boldsymbol{b}$$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}$$

subject to an incompressibility constraint:

$$\frac{dJ}{dt} = 0 \Longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

- Traditionally, $\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x}$ and the formulation **locks** [Gil, Arranz, Bonet, 2010]
- Introduce additional Lagrangian equation for $\boldsymbol{F}$ [Gil, Arranz, Bonet, 2013]:

$$\frac{d\boldsymbol{F}}{dt} = \boldsymbol{\nabla}_0 \boldsymbol{v} \Longrightarrow \frac{d\boldsymbol{F}}{dt} = (\boldsymbol{\nabla} \boldsymbol{v}) \boldsymbol{F}$$

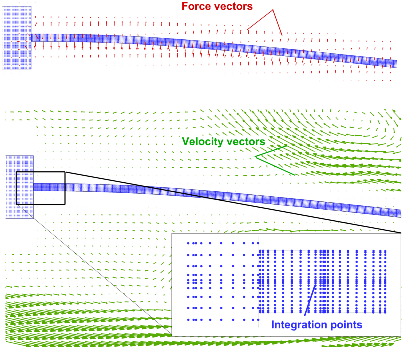
- Similar structure to that of $\boldsymbol{v}$-$\boldsymbol{F}$-J **mixed solid formulation**

Key ingredients of the ISPM

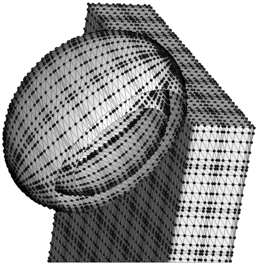
- A **fast** background fluid solver.
- A way to spatially interpolate the kinematics (i.e. $\boldsymbol{v}$, $\boldsymbol{l}$) from the background fluid to the foreground structure: **kernel functions** φ.
- Structure (or other physics) introduced by means of an immersed potential sampled at a collection of **integration points** a_p.

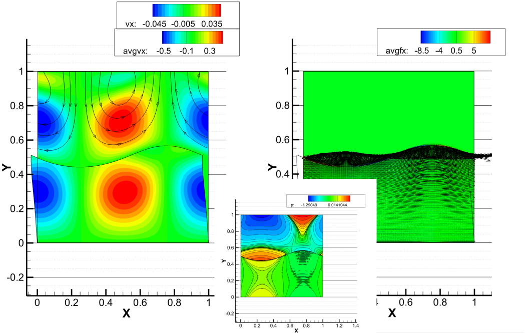
$$\Pi^s(\phi) = \int_{\Omega_0^s} \widehat{\Psi}^s(\phi)\, dV \simeq \sum_{a_p} \widehat{\Psi}^s\left(\phi^{a_p}\right) w^{a_p}$$

- Computation of **FSI forces** carried out from the directional derivative of the energy functional: spatial integration of structure stresses.
- Computation of structure stresses requires prior knowledge of the deformation gradient tensor $\boldsymbol{F}$.



Some FSI simulations





[MOVIE 2D constant flow and cell interacting with membranes]

[MOVIE 2D pulsatile flow and cell interacting with membrane]

[MOVIE 2D FSI rotating rigid object]

[MOVIE 2D rotating rigid object and flexible beam]

[MOVIE 3D pulsatile flow and cells interacting with membranes]

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Concluding remarks

Conclusions

- **First order conservation laws** in Total Lagrangian settting
- **Adapted** CFD discretisation techniques can be used:
  - Cell Centred FVM
  - Vertex Centred FVM
  - Taylor Galerkin FEM
  - Petrov Galerkin FEM
  - Fractional step integration for incompressible materials
- Stresses converge at the **same rate** as velocities and displacements
- Linear tetrahedra can be used without numerical **instabilities**:
  - Bending difficulties
  - Volumetric locking
  - Pressure instabilites
- Standard CFD techniques for **shock** capturing can be applied

On-going work

- Implement in open-source OpenFOAM
- Updated Lagrangian and ALE formulation
- Monolithic FSI
- Fracture, contact and biomedical applications
- SPH and DG frameworks

Journal publications

Fast solid dynamics

- C. H. Lee, A. J. Gil and J. Bonet. *Development of a cell centred upwind finite volume algorithm for a new conservation law formulation in structural dynamics*, Computers & Structures 118 (2013) 13-38.
- I. A. Karim, C. H. Lee, A. J. Gil and J. Bonet. *A Two-Step Taylor Galerkin formulation for fast dynamics*, Engineering Computations 31 (2014) 366-387.
- C. H. Lee, A. J. Gil and J. Bonet. *Development of a stabilised Petrov-Galerkin formulation for a mixed conservation law formulation in fast solid dynamics*, CMAME 268 (2014) 40-64.
- M. Aguirre, A. J. Gil, J. Bonet and A. A. Carreño. *A vertex centred Finite Volume Jameson-Schmidt-Turkel (JST) algorithm for a mixed conservation formulation in solid dynamics*, JCP 259 (2014) 672-699.
- A. J. Gil, C. H. Lee, J. Bonet and M. Aguirre. *A stabilised Petrov-Galerkin formulation for linear tetrahedral elements in compressible, nearly incompressible and truly incompressible fast dynamics*, CMAME 276 (2014) 659-690.
- J. Bonet, A. J. Gil, C. H. Lee, M. Aguirre and R. Ortigosa. *A first order hyperbolic framework for large strain computational solid dynamics. Part I: Total Lagrangian Isothermal Elasticity*, CMAME 283 (2015) 689-732.
- M. Aguirre, A. J. Gil, J. Bonet and C. H. Lee. *A vertex centred upwind finite volume method for solid dynamics*, JCP. In Press.
- C. H. Lee, A. J. Gil, J. Bonet and R. Ortigosa. *A Total Lagrangian Hydrocode for linear tetrehedral elements in compressible, nearly incompressible and truly incompressible fast solid dynamics*, CMAME. Under review.

Fluid structure interaction

- A. J. Gil, A. A. Carreño, J. Bonet and O. Hassan, *The Immersed Structural Potential Method for haemodynamic applications*. JCP 229 (2010) 8613-8641.
- C. Hesch, A. J. Gil, A. A. Carreño and J. Bonet. *On continuum immersed strategies for Fluid-Structure Interaction*, CMAME 247-248 (2012) 51-64.
- A. J. Gil, A. A. Carreño, J. Bonet and O. Hassan, *An enhanced Immersed Structural Potential Method*. JCP 250 (2013) 178-205.
- C. Hesch, A. J. Gil, A. A. Carreño, J. Bonet and P. Betsch, *A mortar approach for Fluid-Structure Interaction problems: Immersed strategies for deformable and rigid bodies*. CMAME 278 (2014) 853-882.

In preparation

- J. Bonet, A. J. Gil and C. H. Lee, *A first order hyperbolic framework for large strain computational solid dynamics. Part II: Non-isothermal and plasticity*, CMAME.
- A. A. Carreño and A. J. Gil, *RKCP-ISPM for Fluid- Structure Interaction*, JCP.

Acknowledgements

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