Micro-mechanical solutions and their use in constitutive models for cementitious materials

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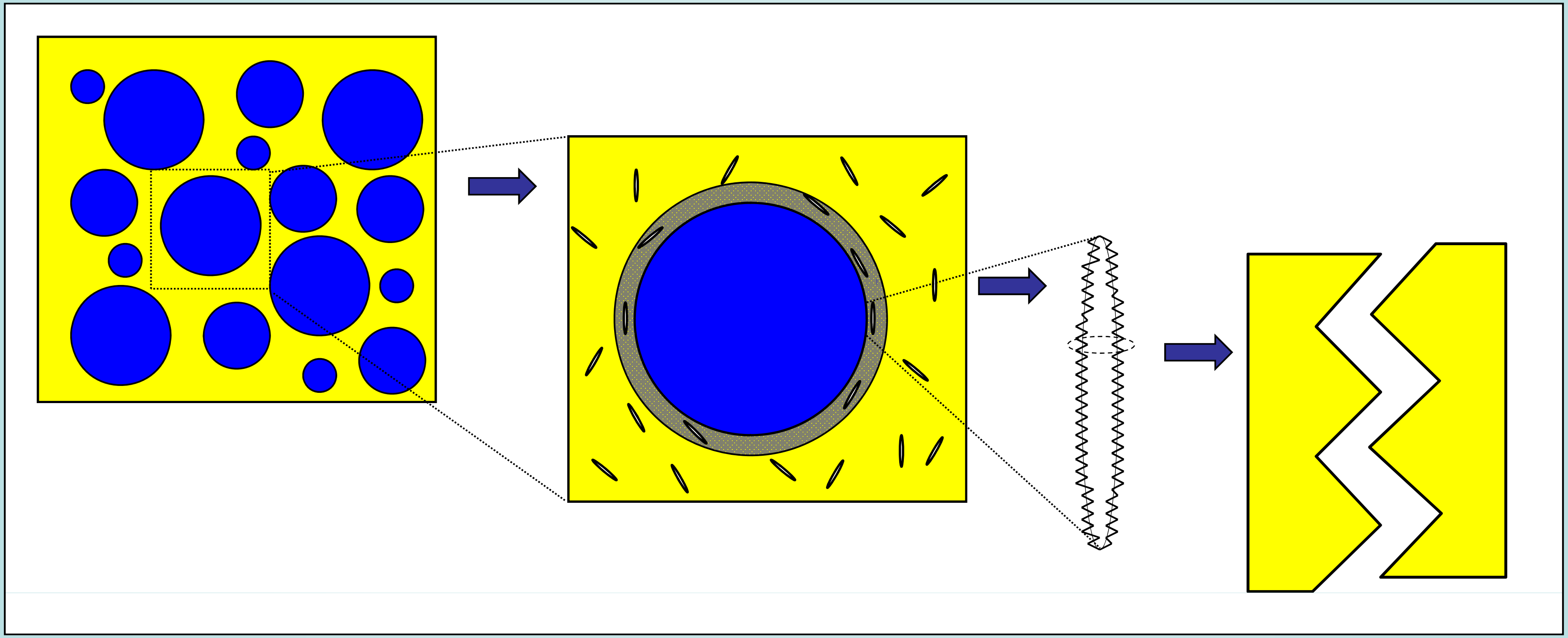

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Introduction & Motivation

Macroscopic models

- Complex empirical functions
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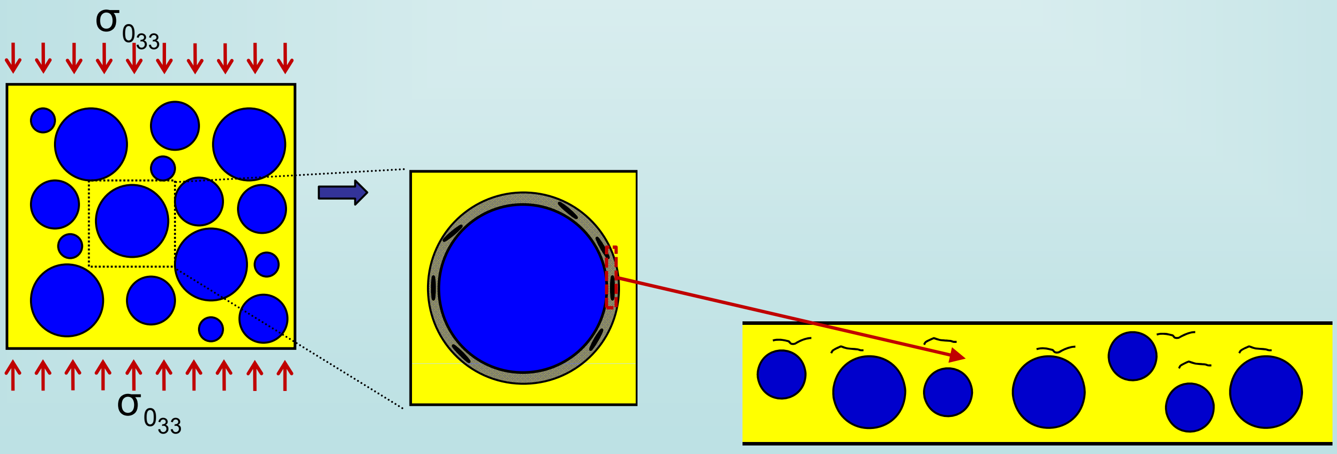
Micromechanical models

- Built on (relatively) simple micro-mechanical solutions
- Use physically meaningful parameters
- Reliable over a full range of responses
- More accurate and tractable (in theory)

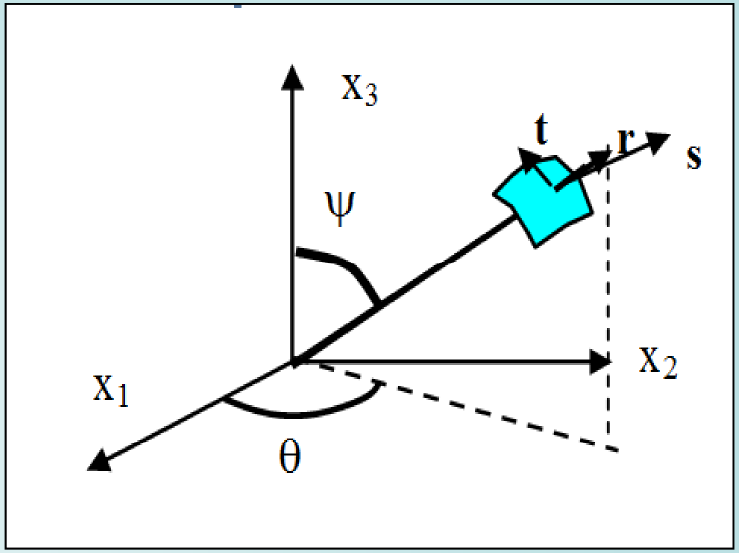


Part 1. Micro-cracked solids *(Budiansky & O'Connell, 1976)*

The crack opening displacements at a radial coordinate a_0 for a circular crack of radius a_{0i} in an infinite isotropic elastic body are given by;

$$\mathbf{u}_{rst}(a) = \alpha_{u}(a)\begin{bmatrix} \sigma_{rr} \\ \dfrac{2}{2-\nu}\sigma_{rs} \\ \dfrac{2}{2-\nu}\sigma_{rt} \end{bmatrix} \qquad (1)$$

where $\alpha_u(a) = \dfrac{8\left(1-\nu^2\right)}{\pi E}\sqrt{{a_{0_i}}^2 - a^2}$



If the crack area is denoted by A_c , the associated additional strains for crack i are given by;

$$\boldsymbol{\varepsilon}_{rst\_add_i} = \frac{1}{{a_i}^3}\int_{A_{c_i}} \mathbf{u}_{rst}\, dA_c \qquad (2)$$

The associated Cartesian added strains are;

$$\boldsymbol{\varepsilon}_{add_i} = \frac{1}{{a_{0i}}^3}\int_{A_{c_i}} \frac{1}{2}\left(\mathbf{r}\otimes\mathbf{u}_{rst} + \mathbf{u}_{rst}\otimes\mathbf{r}\right) dA_c$$

Micro-cracked solid

The non-zero additional strain tensor components from a dilute series of cracks with the same normal **r** are given by;

$$\boldsymbol{\varepsilon}_{\alpha} = \begin{bmatrix} \varepsilon_{\alpha rr} \\ \varepsilon_{\alpha rs} \\ \varepsilon_{\alpha rt} \end{bmatrix} = f\,\frac{16\,(1\text{-}\nu^2)}{3E} \begin{bmatrix} \sigma_{rr} \\ \dfrac{2}{2-\nu}\sigma_{rs} \\ \dfrac{2}{2-\nu}\sigma_{rt} \end{bmatrix} \qquad (3)$$

in which $f = N\, a_0^3$.

f is the crack density parameter of Budiansky and O'Connell [16], N is the number of cracks per unit volume and a_0 is a crack radius.

Equation (3) may be written (in matrix notation):

$$\boldsymbol{\varepsilon}_{\alpha} = f\ \mathbf{C}_{ac} : \mathbf{s} \qquad (4)$$

where $\mathbf{s}$ = stress tensor & C_{ac} contains elastic compliance terms from (3)

Micro-cracked solid

In general micro-cracks occur in more than one direction and to calculate the total additional strain (ε_a) in the reference configuration it is necessary to transform and sum the contributions from all directions.

If the crack density parameter is considered to be a function of direction i.e. $f=func(\theta,\psi)$, then the additional strain for a set of n_a discrete directions is given by equation (5) and for a continuous distribution of micro-cracks by equation (6)

$$\boldsymbol{\varepsilon}_a = \sum_{n_a} \mathbf{N}_\varepsilon : \boldsymbol{\varepsilon}_{\alpha i} \tag{5}$$

$$\boldsymbol{\varepsilon}_a = \frac{1}{2\pi} \int_{2\pi} \int_{\frac{\pi}{2}} \mathbf{N}_\varepsilon : \boldsymbol{\varepsilon}_\alpha(\theta,\psi) \sin(\psi)\, \mathrm{d}\psi\, \mathrm{d}\theta \tag{6}$$

Defining the stress transformation

$$\mathbf{s} = \mathbf{N} : \boldsymbol{\sigma} \tag{7}$$

Using (4) to (7), the total additional strain may be obtained as:

$$\boldsymbol{\varepsilon}_a = \left(\frac{1}{2\pi} \int_{2\pi} \int_{\frac{\pi}{2}} \mathbf{N}_\varepsilon \cdot \mathbf{C}_{ac} \cdot \mathbf{N} f \sin(\psi)\, \mathrm{d}\psi\, \mathrm{d}\theta \right) : \boldsymbol{\sigma} \tag{8}$$

Micro-cracked solid

The stress elastic strain relationship may be written

$$\boldsymbol{\sigma} = \mathbf{D}_{el} : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_a\right) \tag{9}$$

Using (8) in (9) and rearranging gives

$$\boldsymbol{\sigma} = \left(\mathbf{I}^{4s} + \frac{1}{2\pi} \cdot \mathbf{D}_{el} \cdot \int_{2\pi} \int_{\frac{\pi}{2}} \mathbf{N}_{\varepsilon} \cdot \mathbf{C}_{ac} \cdot \mathbf{N} f(\theta,\psi) \sin(\psi)\, \mathrm{d}\psi \, \mathrm{d}\theta \right)^{-1} \cdot \mathbf{D}_{el} : \boldsymbol{\varepsilon} \tag{10}$$

If there are a finite number of micro-cracking directions n_a, (10) becomes

$$\boldsymbol{\sigma} = \left(\mathbf{I}^{4\mathrm{s}} + \mathbf{D}_{\mathrm{el}} \cdot \sum_{\mathrm{n_a}} \mathbf{N}_{\varepsilon} \cdot \mathbf{C}_{\mathrm{ac}} \cdot \mathbf{N} f_{\mathrm{i}} \right)^{-1} \cdot \mathbf{D}_{\mathrm{el}} : \boldsymbol{\varepsilon} \tag{11}$$

It is convenient to replace f with a micro-cracking parameter $\omega \in [0, 1]$ such that

$$f = \frac{3}{16\left(1-\upsilon^2\right)} \left(\frac{\omega}{1-\omega} \right) \tag{12}$$

Final expressions in matrix notation

Although many papers, and the preceding derivation, are written in direct tensor notation, most FE codes use engineering shear strains and matrices (or equivalent vectors). Making use of equation (12), and switching to matrix notation, the relationships in (10) and (11) become:

$$\boldsymbol{\sigma} = \left(\mathbf{I} + \frac{1}{2\pi} \mathbf{D}_{el} \int_{2\pi} \int_{\frac{\pi}{2}} \mathbf{N}_{\varepsilon}{}^{\mathrm{T}} \mathbf{C}_L \mathbf{N} \frac{\omega(\theta,\psi)}{1-\omega(\theta,\psi)} \sin(\psi)\, \mathrm{d}\psi \; \mathrm{d}\theta \right)^{-1} \mathbf{D}_{el}\, \boldsymbol{\varepsilon} \qquad (13)$$

$$\boldsymbol{\sigma} = \left(\mathbf{I} + \mathbf{D}_{\mathrm{el}} \sum_{\mathrm{n_a}} \mathbf{N}_{\varepsilon}{}^{\mathrm{T}} \mathbf{C}_{\mathrm{L}} \mathbf{N} \left(\frac{\omega_{\mathrm{i}}}{1-\omega_{\mathrm{i}}} \right) \right)^{-1} \mathbf{D}_{\mathrm{el}}\, \boldsymbol{\varepsilon} \qquad (14)$$

in which

$$C_L = \frac{1}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{4}{2-\nu} & 0 \\ 0 & 0 & \dfrac{4}{2-\nu} \end{bmatrix}$$

Direct tensor and matrix representations

Stresses are directly equivalent in matrix and direct tensor notation whereas strains are different, as shown below.

$$\boldsymbol{\sigma}_{matrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \boldsymbol{\sigma}_{dir\_tensor} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_{matrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \quad \text{whereas } \boldsymbol{\varepsilon}_{dir\_tensor} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}$$

Examples

The examples consider a 2D plane-stress body containing micro-cracks

The two examples presented show the effect of increasing a reference ω_{max}

(i) A continuous variation of micro-cracks → $\omega(\theta)=\omega_{max}(1-\sin(\theta))$

(ii) Two sets of micro-cracks.

The model was programmed in Mathcad

Example 1. Plane stress, evenly distributed microcracks

$E := 40000$ **N/mm²** **Young's modulus** $\upsilon := 0.2$ **Poisson's ratio** $ORIGIN := 1$

Elastic and added compliance matrices

$$C_L := \frac{1}{E}\cdot\begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{2-\upsilon} \end{pmatrix} \qquad I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D_{el} := \frac{E}{1-\upsilon^2}\cdot\begin{pmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{pmatrix}$$

Transformation matrices

$$Ns(\theta) := \begin{bmatrix} (\cos(\theta))^2 & (\sin(\theta))^2 & 2\cdot\cos(\theta)\cdot\sin(\theta) \\ -\cos(\theta)\cdot\sin(\theta) & \sin(\theta)\cos(\theta) & \cos(\theta)^2 - \sin(\theta)^2 \end{bmatrix} \qquad Ne(\theta) := \begin{bmatrix} (\cos(\theta))^2 & (\sin(\theta))^2 & \cos(\theta)\cdot\sin(\theta) \\ -2\cdot\cos(\theta)\cdot\sin(\theta) & 2\cdot(\sin(\theta)\cos(\theta)) & \cos(\theta)^2 - \sin(\theta)^2 \end{bmatrix}$$

Constitutive matrices

$$\omega(\theta,\omega_{max}) := \omega_{max}\cdot(1-\sin(\theta))$$

Vary microcracking from ? $_{max}$ at ?=0 to 1 at ?=p/2

$$dNC\omega N(\theta,\omega_{max}) := Ne(\theta)^T\cdot C_L\cdot\frac{\omega(\theta,\omega_{max})}{(1-\omega(\theta,\omega_{max}))}\cdot Ns(\theta)$$

$$NC\omega N(\omega_{max}) := \frac{1}{\pi}\cdot\begin{pmatrix} \int_0^\pi dNC\omega N(\theta,\omega_{max})_{1,1}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{1,2}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{1,3}\,d\theta \\ \int_0^\pi dNC\omega N(\theta,\omega_{max})_{2,1}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{2,2}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{2,3}\,d\theta \\ \int_0^\pi dNC\omega N(\theta,\omega_{max})_{3,1}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{3,2}\,d\theta & \int_0^\pi dNC\omega N(\theta,\omega_{max})_{3,3}\,d\theta \end{pmatrix}$$

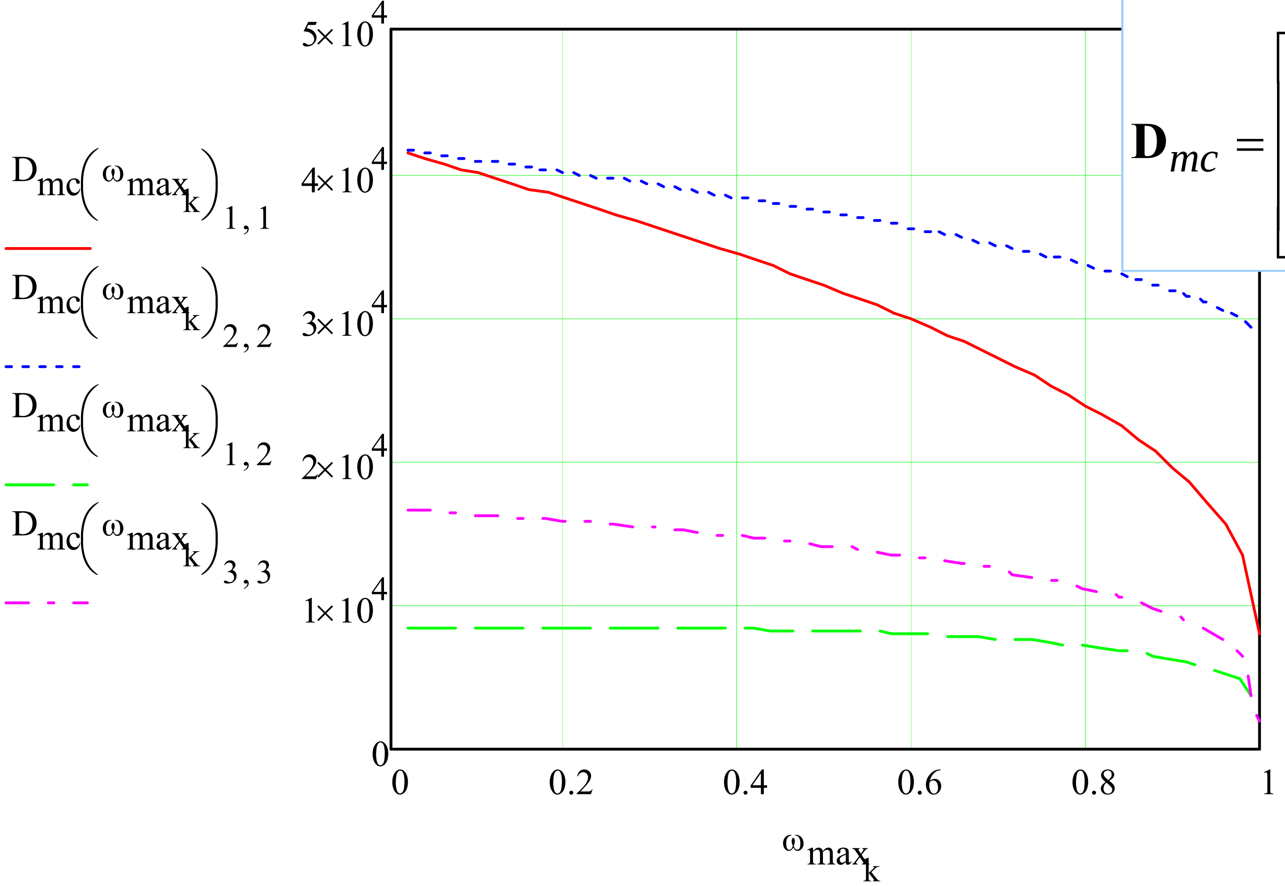
$$D_{mc}(\omega_{max}) := \left(I + D_{el}\cdot NC\omega N(\omega_{max})\right)^{-1}\cdot D_{el}$$

Example 1. Plane stress, evenly distributed microcracks

$n := 50 \qquad k := 1..n \qquad \omega_{max_k} := \frac{k}{n}\cdot\left(1 - 10^{-8}\right)$

$\boldsymbol{\sigma} = \mathbf{D}_{mc}\boldsymbol{\varepsilon}$

$$\mathbf{D}_{mc} = \begin{bmatrix} D_{mc1,1} & D_{mc1,2} & D_{mc1,3} \\ D_{mc2,1} & D_{mc2,2} & D_{mc2,3} \\ D_{mc3,1} & D_{mc3,2} & D_{mc3,3} \end{bmatrix}$$



Example 2. Plane stress, uni-orientated microcracks

Properties and transformation matrices as per example 1

Constitutive matrices

$$dNCN(\theta) := Ne(\theta)^T \cdot C_L \cdot Ns(\theta)$$

$n_c := 2$ **Align the micro-crack normals with the x and 35° to x axis**

$$\theta := \begin{pmatrix} 0 \\ 35 \cdot \frac{\pi}{180} \end{pmatrix}$$

$$NC\omega N(\omega) := \sum_{i=1}^{n_c} \left[\frac{\omega_i}{(1-\omega_i)} \cdot Ne(\theta_i)^T \cdot C_L \cdot Ns(\theta_i) \right]$$

Noting that ? is now a array of values

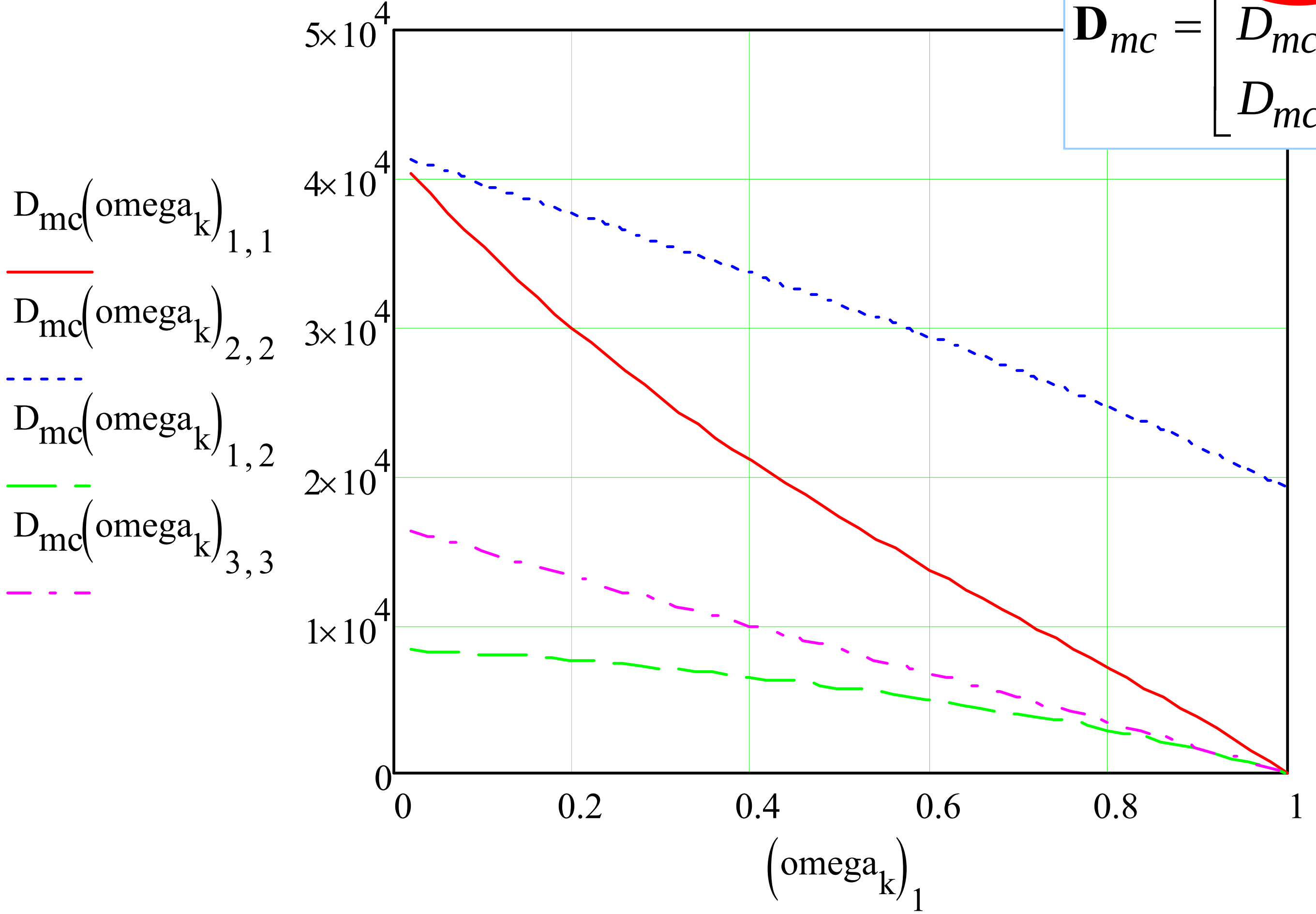
$$D_{mc}(\omega) := \left(I + D_{el} \cdot NC\omega N(\omega)\right)^{-1} \cdot D_{el}$$

Example 2

$n := 50 \qquad k := 1 .. n \qquad omega_k := \begin{pmatrix} 0.999999 \\ 0.5 \end{pmatrix} \cdot \frac{k}{n}$

$\boldsymbol{\sigma} = \mathbf{D}_{mc}\boldsymbol{\varepsilon}$

$$\mathbf{D}_{mc} = \begin{bmatrix} D_{mc1,1} & D_{mc1,2} & D_{mc1,3} \\ D_{mc2,1} & D_{mc2,2} & D_{mc2,3} \\ D_{mc3,1} & D_{mc3,2} & D_{mc3,3} \end{bmatrix}$$



Part 2 . Eshelby solution *(Eshelby, 1957)*

If an elastic ellipsoidal sub-region in an infinite elastic domain undergoes a change in size and/or shape (i.e. a transformation) the state of stress in the sub-region is uniform.

If the change in strain in the sub-region in a stress free state is denoted by the '**eigenstrain**' tensor (ε_t) then the constrained strain in the sub-region (ε_c**)** is given by

$$\boldsymbol{\varepsilon}_c = \mathbf{S} : \boldsymbol{\varepsilon}_t \qquad (1)$$

S is the Eshelby tensor; **S** for a spherical isotropic inclusions has the following non-zero tensor terms

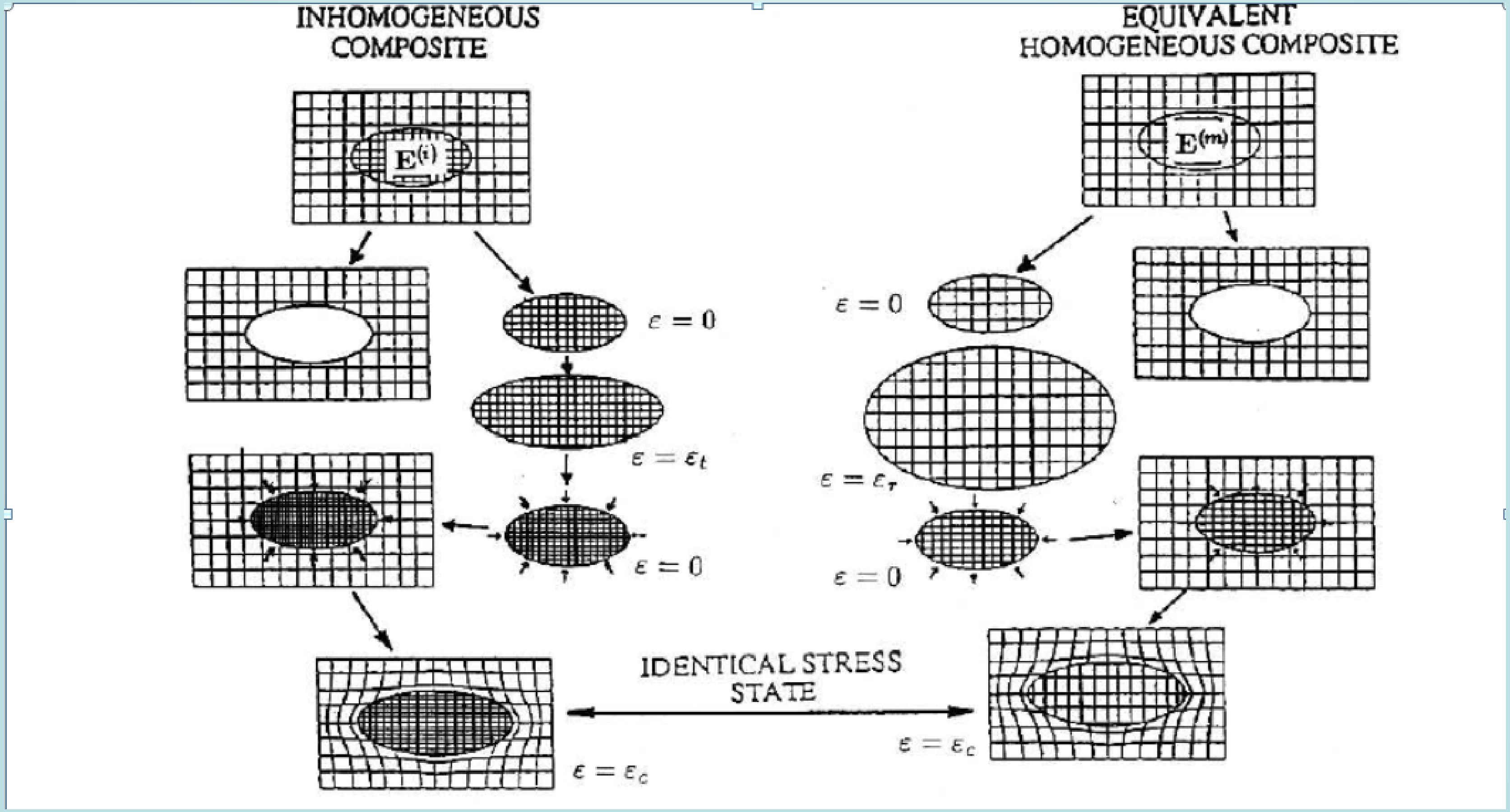
$$S_{ijkl} = \frac{5\upsilon - 1}{15(1-\upsilon)}\delta_{ij}\delta_{kl} + \frac{4-5\upsilon}{15(1-\upsilon)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

in which δ_{ij} is the Kronecker delta

Eshelby solution

Inhomogeneous inclusions.

The inclusion is replaced with an equivalent inclusion made of the matrix material in such a way that the constrained strain and stress are the same as for the inhomogeneous inclusion. The stress-free (eigenstrain) strain in the equivalent inclusion is denoted ε_τ. The equivalence is well illustrated in the diagram below (taken from Withers et al 1989.)



Eshelby solution

Notation

D_Ω and **D_m** = elasticity tensors of inclusions and matrix resp.

σ_Ω and σ_m = stresses for the inclusions and matrix resp.

Solution with far-field strain ε_0

ε_τ (transformation strain) is now only used to account for material mismatch

$$\boldsymbol{\varepsilon}_\Omega = \boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_c = \boldsymbol{\varepsilon}_0 + \mathbf{S} : \boldsymbol{\varepsilon}_\tau \quad (2)$$

The stress in the inclusion is given by

$$\boldsymbol{\sigma}_\Omega = \mathbf{D}_\Omega : (\boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_c) \quad (3)$$

ε_τ is used to achieve the 'equivalent' stress, as follows;

$$\mathbf{D}_\Omega : (\boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_c) = \mathbf{D}_m : (\boldsymbol{\varepsilon}_0 + \boldsymbol{\varepsilon}_c - \boldsymbol{\varepsilon}_\tau) \quad (4)$$

Using equation (2), equation (4) becomes;

$$(\mathbf{D}_\Omega - \mathbf{D}_m) : \boldsymbol{\varepsilon}_0 + ((\mathbf{D}_\Omega - \mathbf{D}_m) \cdot \mathbf{S} + \mathbf{D}_m) : \boldsymbol{\varepsilon}_\tau = \mathbf{0} \quad (5)$$

from which;

$$\boldsymbol{\varepsilon}_\tau = \mathbf{A}_\Omega : \boldsymbol{\varepsilon}_0 \quad (6)$$

where $\mathbf{A}_\Omega = ((\mathbf{D}_\Omega - \mathbf{D}_m) \cdot \mathbf{S} + \mathbf{D}_m)^{-1} \cdot (\mathbf{D}_\Omega - \mathbf{D}_m)$

Eshelby solution

Using (6) in (2) and the result in (3) gives the following;

$$\boldsymbol{\sigma}_{\Omega} = \mathbf{D}_{\Omega} \cdot \left(\mathbf{I}^{4s} + \mathbf{S} \cdot \mathbf{A}_{\Omega} \right) : \boldsymbol{\varepsilon}_0 \qquad (7)$$

$$\boldsymbol{\sigma}_{\Omega} = \mathbf{D}_{m} \cdot \left[\mathbf{I}^{4s} + (\mathbf{S} - \mathbf{I}^{4s}) \cdot \mathbf{A}_{\Omega} \right] : \boldsymbol{\varepsilon}_0 \qquad (8)$$

Volume averages, Mori-Tanaka and mean stress-strain relationships

If f_m = volume of matrix material and f_Ω = volume of inclusions then

$$f_m + f_\Omega = 1 \qquad (9)$$

For the individual phases

$$\boldsymbol{\sigma}_{\Omega} = \mathbf{D}_{\Omega} : \boldsymbol{\varepsilon}_{\Omega} \qquad (10a)$$

$$\boldsymbol{\sigma}_{\mathrm{m}} = \mathbf{D}_{\mathrm{m}} : \boldsymbol{\varepsilon}_{\mathrm{m}} \qquad (10b)$$

Comparing (7) and (11) leads to;

$$\boldsymbol{\varepsilon}_{\Omega} = \left(\mathbf{I}^{4s} + \mathbf{S} \cdot \mathbf{A}_{\Omega} \right) : \boldsymbol{\varepsilon}_0 \qquad (11)$$

In the **Mori-Tanaka method,** the argument is made that the when the inclusions are not dilute the ‘disturbance’ strain may be based on the average matrix stresses/strains rather than the far field stress (or strain).

Eshelby solution

Thus (11) becomes;

$$\boldsymbol{\varepsilon}_{\Omega} = \mathbf{T}_{\Omega} : \boldsymbol{\varepsilon}_{m} \qquad (12)$$

in which $\mathbf{T}_{\Omega} = \mathbf{I}^{4s} + \mathbf{S} \cdot \mathbf{A}_{\Omega}$

The mean stress is given by;

$$\bar{\boldsymbol{\sigma}} = \mathrm{f}_{\Omega} \boldsymbol{\sigma}_{\Omega} + \mathrm{f}_{\mathrm{m}} \boldsymbol{\sigma}_{\mathrm{m}} \qquad (13)$$

and the mean strain by;

$$\bar{\boldsymbol{\varepsilon}} = \mathrm{f}_{\Omega} \boldsymbol{\varepsilon}_{\Omega} + \mathrm{f}_{\mathrm{m}} \boldsymbol{\varepsilon}_{\mathrm{m}} \qquad (14)$$

From (12) and (14)

$$\boldsymbol{\varepsilon}_{\mathrm{m}} = \left(\mathrm{f}_{\Omega} \mathbf{T}_{\Omega} + \mathrm{f}_{\mathrm{m}}\right)^{-1} : \bar{\boldsymbol{\varepsilon}} \qquad (15)$$

Then the mean stress can be obtained from (15) and (13) to be;

$$\bar{\boldsymbol{\sigma}} = \left(f_{\Omega}\, \mathbf{D}_{\Omega} \cdot \mathbf{T}_{\Omega} + f_{m} \mathbf{D}_{m} \right) : \boldsymbol{\varepsilon}_{m} \qquad (16)$$

The means stress-strain relationship is;

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}_{\mathrm{m}\Omega} : \bar{\boldsymbol{\varepsilon}} \qquad (17)$$

where $\mathbf{D}_{m\Omega} = \left(f_{\Omega} \mathbf{D}_{\Omega} \cdot \mathbf{T}_{\Omega} + f_{m} \mathbf{D}_{m} \right) \cdot \left(f_{\Omega} \mathbf{T}_{\Omega} + f_{m} \mathbf{I}^{4s} \right)^{-1}$

Eshelby Example

ESHELBY TWO PHASE COMPOSITE: MATRICES. MATRIX NOTATION

$ORIGIN := 1$

Material parameters

$E_m := 30000$ **N/mm²** **Young's modulus of matrix**

$\upsilon_m := 0.15$ **Poisson's ratio of matrix**

$E_\Omega := 70000$ **N/mm²** **Young's modulus of inclusion**

$\upsilon_\Omega := 0.29$ **Poisson's ratio of inclusions**

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$princ(\sigma) := reverse\left[sort\left[eigenvals\left(\begin{pmatrix} \sigma_1 & \sigma_4 & \sigma_5 \\ \sigma_4 & \sigma_2 & \sigma_6 \\ \sigma_5 & \sigma_6 & \sigma_3 \end{pmatrix}\right)\right]\right]$$

Volumetric proportions:

$f_\Omega := 0.75 \quad f_m := 1 - f_\Omega$

Elastic D-matrix:

$$\alpha1_m := \frac{\upsilon_m}{1-\upsilon_m} \qquad \alpha2_m := \frac{1-2\cdot\upsilon_m}{2\cdot(1-\upsilon_m)} \qquad \alpha1_\Omega := \frac{\upsilon_\Omega}{1-\upsilon_\Omega} \qquad \alpha2_\Omega := \frac{1-2\cdot\upsilon_\Omega}{2\cdot(1-\upsilon_\Omega)}$$

$$D_m := \left[E_m\cdot\frac{1-\upsilon_m}{(1+\upsilon_m)\cdot(1-2\cdot\upsilon_m)}\right]\cdot\begin{pmatrix} 1 & \alpha1_m & \alpha1_m & 0 & 0 & 0 \\ \alpha1_m & 1 & \alpha1_m & 0 & 0 & 0 \\ \alpha1_m & \alpha1_m & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha2_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha2_m & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha2_m \end{pmatrix}$$

$$D_\Omega := \left[E_\Omega\cdot\frac{1-\upsilon_\Omega}{(1+\upsilon_\Omega)\cdot(1-2\cdot\upsilon_\Omega)}\right]\cdot\begin{pmatrix} 1 & \alpha1_\Omega & \alpha1_\Omega & 0 & 0 & 0 \\ \alpha1_\Omega & 1 & \alpha1_\Omega & 0 & 0 & 0 \\ \alpha1_\Omega & \alpha1_\Omega & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha2_\Omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha2_\Omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha2_\Omega \end{pmatrix}$$

Example

Set the Eshelby matrix and compute constitutive matrices

$$s1 := \frac{7 - 5\cdot\upsilon_m}{15\cdot(1-\upsilon_m)} \qquad s2 := \frac{5\cdot\upsilon_m - 1}{15\cdot(1-\upsilon_m)} \qquad s3 := \frac{4 - 5\cdot\upsilon_m}{15\cdot(1-\upsilon_m)}$$

$$S_\Omega := \begin{pmatrix} s1 & s2 & s2 & 0 & 0 & 0 \\ s2 & s1 & s2 & 0 & 0 & 0 \\ s2 & s2 & s1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\cdot s3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\cdot s3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\cdot s3 \end{pmatrix} \qquad S_\Omega = \begin{pmatrix} 0.490196 & -0.019608 & -0.019608 & 0 & 0 & 0 \\ -0.019608 & 0.490196 & -0.019608 & 0 & 0 & 0 \\ -0.019608 & -0.019608 & 0.490196 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.509804 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.509804 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.509804 \end{pmatrix}$$

$$A_\Omega := \left[(D_\Omega - D_m)\cdot S_\Omega + D_m\right]^{-1}\cdot(D_m - D_\Omega)$$

$$T_\Omega := I + S_\Omega\cdot A_\Omega$$

$$D_{m\Omega} := (f_\Omega\cdot D_\Omega\cdot T_\Omega + f_m\cdot D_m)\cdot(f_\Omega\cdot T_\Omega + f_m\cdot I)^{-1} \qquad W_{m\Omega} := D_m\cdot(f_\Omega\cdot D_\Omega\cdot T_\Omega + f_m\cdot D_m)^{-1}$$

$$D_{m\Omega} = \begin{pmatrix} 67408.4 & 22746.1 & 22746.1 & 0 & 0 & 0 \\ 22746.1 & 67408.4 & 22746.1 & 0 & 0 & 0 \\ 22746.1 & 22746.1 & 67408.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22331.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22331.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22331.2 \end{pmatrix}$$

Example

Alternative form (only suitable for isotropic cases)

Matrices in terms of G, K

$$s_I := \frac{1+\upsilon_m}{3\cdot(1-\upsilon_m)} \qquad s_I = 0.45098 \qquad s_{II} := \frac{2\cdot(4-5\cdot\upsilon_m)}{15\cdot(1-\upsilon_m)} \qquad s_{II} = 0.509804$$

$$K_m := \frac{E_m}{3\cdot(1-2\cdot\upsilon_m)} \qquad G_m := \frac{E_m}{2\cdot(1+\upsilon_m)} \qquad K_\Omega := \frac{E_\Omega}{3\cdot(1-2\cdot\upsilon_\Omega)} \qquad G_\Omega := \frac{E_\Omega}{2\cdot(1+\upsilon_\Omega)}$$

$$K_{m\Omega} := \frac{K_m\cdot\left[K_\Omega\cdot\left[f_\Omega+s_I\cdot(1-f_\Omega)\right]+K_m\cdot(1-s_I)\cdot(1-f_\Omega)\right]}{\left[(1-f_\Omega)\cdot s_I\cdot K_\Omega\right]+K_m\cdot\left[1-s_I\cdot(1-f_\Omega)\right]} \qquad G_{m\Omega} := \frac{G_m\cdot\left[G_\Omega\cdot\left[f_\Omega+s_{II}\cdot(1-f_\Omega)\right]+G_m\cdot\left[(1-s_{II})\cdot(1-f_\Omega)\right]\right]}{\left[(1-f_\Omega)\cdot s_{II}\cdot G_\Omega\right]+G_m\cdot\left[1-s_{II}\cdot(1-f_\Omega)\right]}$$

$$E_{m\Omega} := \frac{9\cdot K_{m\Omega}\cdot G_{m\Omega}}{3\cdot K_{m\Omega}+G_{m\Omega}} \qquad E_{m\Omega} = 55930.680766 \qquad \upsilon_{m\Omega} := \frac{3\cdot K_{m\Omega}-E_{m\Omega}}{6\cdot K_{m\Omega}} \qquad \upsilon_{m\Omega} = 0.252301 \qquad \frac{K_{m\Omega}}{K_m} = 2.634347 \qquad \frac{G_{m\Omega}}{G_m} = 1.712056$$

$$\alpha1_{m\Omega} := \frac{\upsilon_{m\Omega}}{1-\upsilon_{m\Omega}} \qquad \alpha2_{m\Omega} := \frac{1-2\cdot\upsilon_{m\Omega}}{2\cdot(1-\upsilon_{m\Omega})}$$

$$D_{m\Omega} := \left[E_{m\Omega}\cdot\frac{1-\upsilon_{m\Omega}}{(1+\upsilon_{m\Omega})\cdot(1-2\cdot\upsilon_{m\Omega})}\right]\cdot\begin{pmatrix} 1 & \alpha1_{m\Omega} & \alpha1_{m\Omega} & 0 & 0 & 0 \\ \alpha1_{m\Omega} & 1 & \alpha1_{m\Omega} & 0 & 0 & 0 \\ \alpha1_{m\Omega} & \alpha1_{m\Omega} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha2_{m\Omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha2_{m\Omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha2_{m\Omega} \end{pmatrix} = \begin{pmatrix} 67408.4 & 22746.1 & 22746.1 & 0 & 0 & 0 \\ 22746.1 & 67408.4 & 22746.1 & 0 & 0 & 0 \\ 22746.1 & 22746.1 & 67408.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22331.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22331.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22331.2 \end{pmatrix}$$

Example

MATRIX AND INCLUSION STRESSES FOR FOR VARIOUS CASES

Uniaxial compression

$$\sigma_{mean} := \begin{pmatrix} -30 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon_{mean} := D_{m\Omega}^{-1} \cdot \sigma_{mean} \qquad \varepsilon_{mean} = \begin{pmatrix} -5.363782 \times 10^{-4} \\ 1.353289 \times 10^{-4} \\ 1.353289 \times 10^{-4} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\varepsilon_m := \left(f_\Omega \cdot T_\Omega + f_m \cdot I\right)^{-1} \cdot \varepsilon_{mean} = \begin{pmatrix} -0.000764 \\ 0.000151 \\ 0.000151 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon_\Omega := T_\Omega \cdot \varepsilon_m = \begin{pmatrix} -0.00046 \\ 0.00013 \\ 0.00013 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sigma_m := D_m \cdot \varepsilon_m = \begin{pmatrix} -22.51637 \\ 1.36735 \\ 1.36735 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \sigma_\Omega := D_\Omega \cdot \varepsilon_\Omega = \begin{pmatrix} -32.49454 \\ -0.45578 \\ -0.45578 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

Tension + shear

$$\sigma_{mean} := \begin{pmatrix} 5 \\ 0 \\ 1 \\ 2 \\ 1 \\ 4 \end{pmatrix} \qquad \varepsilon_{mean} := D_{m\Omega}^{-1} \cdot \sigma_{mean} \qquad \varepsilon_{mean} = \begin{pmatrix} 8.488541 \times 10^{-5} \\ -2.706577 \times 10^{-5} \\ -4.675535 \times 10^{-6} \\ 8.956095 \times 10^{-5} \\ 4.478047 \times 10^{-5} \\ 1.791219 \times 10^{-4} \end{pmatrix}$$

$$\varepsilon_m := \left(f_\Omega \cdot T_\Omega + f_m \cdot I\right)^{-1} \cdot \varepsilon_{mean} = \begin{pmatrix} 0.00012 \\ -3.02647 \times 10^{-5} \\ 2.5338 \times 10^{-7} \\ 0.00012 \\ 6.10362 \times 10^{-5} \\ 0.00024 \end{pmatrix} \quad \varepsilon_\Omega := T_\Omega \cdot \varepsilon_m = \begin{pmatrix} 7.24053 \times 10^{-5} \\ -2.59995 \times 10^{-5} \\ -6.31851 \times 10^{-6} \\ 7.87238 \times 10^{-5} \\ 3.93619 \times 10^{-5} \\ 0.00016 \end{pmatrix} \quad \sigma_m := D_m \cdot \varepsilon_m = \begin{pmatrix} 3.70715 \\ -0.27347 \\ 0.52265 \\ 1.59225 \\ 0.79612 \\ 3.1845 \end{pmatrix} \quad \sigma_\Omega := D_\Omega \cdot \varepsilon_\Omega = \begin{pmatrix} 5.43095 \\ 0.09116 \\ 1.15912 \\ 2.13592 \\ 1.06796 \\ 4.27183 \end{pmatrix}$$

Principal stresses

$$\mathrm{princ}\left(\sigma_{mean}\right) = \begin{pmatrix} 6.889767 \\ 2.728069 \\ -3.617837 \end{pmatrix} \quad \mathrm{princ}\left(\sigma_m\right) = \begin{pmatrix} 5.211639 \\ 1.898411 \\ -3.153718 \end{pmatrix} \quad \mathrm{princ}\left(\sigma_\Omega\right) = \begin{pmatrix} 7.449144 \\ 3.004622 \\ -3.772543 \end{pmatrix}$$

Part 3. A micro-mechanics-based constitutive model

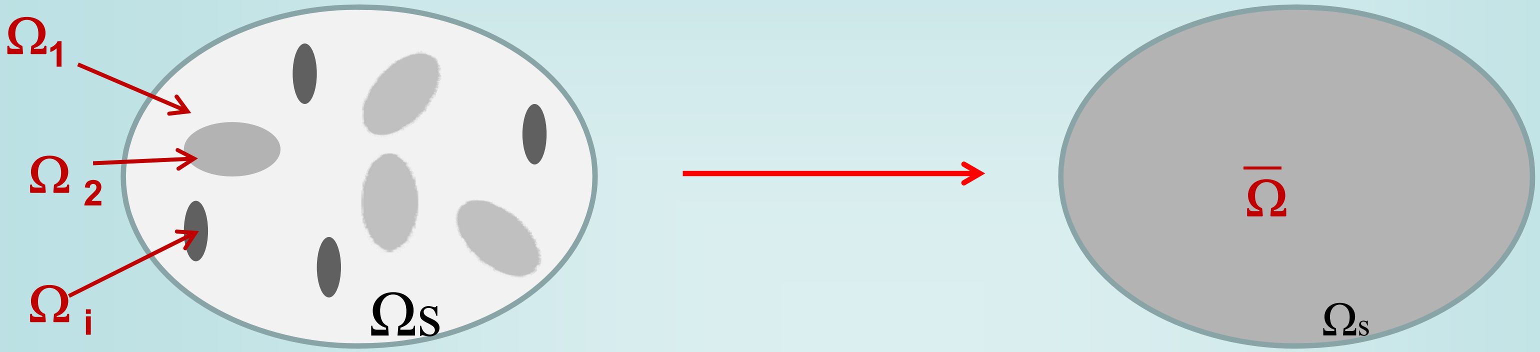
The following slides present the constitutive model described in Mihai & Jefferson, IJSS (2011)

‘A micromechanical model for cementitious composite materials is described in which microcrack initiation, in the interfacial transition zone between aggregate particles and cement matrix, is governed by an exterior-point Eshelby solution. The model assumes a two-phase elastic composite, derived from an Eshelby solution and the Mori-Tanaka homogenization method, to which circular microcracks are added. A multi-component rough crack contact model is employed to simulate normal and shear behaviour of rough microcrack surfaces. ‘

Two-phase composite

Homogenization:

Deriving ‘overall’ or ‘effective’ behaviour (e.g. stiffness, strength properties) from corresponding material behaviour of constituents and from geometrical arrangement of the phases



Homogenization relations:

$$\bar{\boldsymbol{\varepsilon}} = \frac{1}{\Omega_s}\int_{\Omega_s} \boldsymbol{\varepsilon}(x)d\Omega, \quad \bar{\boldsymbol{\sigma}} = \frac{1}{\Omega_s}\int_{\Omega_s} \boldsymbol{\sigma}(x)d\Omega$$

Mean Field Approach:

$$\bar{\boldsymbol{\varepsilon}}_{(p)} = \frac{1}{\Omega_{(p)}}\int_{\Omega_{(p)}} \boldsymbol{\varepsilon}(x)d\Omega, \quad \bar{\boldsymbol{\sigma}}_{(p)} = \frac{1}{\Omega_{(p)}}\int_{\Omega_{(p)}} \boldsymbol{\sigma}(x)d\Omega$$

$$\bar{\boldsymbol{\varepsilon}} = \sum_p \mathrm{f}_{(p)}\bar{\boldsymbol{\varepsilon}}_{(p)}, \quad \bar{\boldsymbol{\sigma}} = \sum_p \mathrm{f}_{(p)}\bar{\boldsymbol{\sigma}}_{(p)}$$

$$\mathrm{f}_{(p)} = \Omega_{(p)} \Big/ \sum_k \Omega_{(k)}$$

Two-phase composite

Average stress and strain for two-phase composite

$$\bar{\boldsymbol{\varepsilon}} = f_{\Omega}\boldsymbol{\varepsilon}_{\Omega} + f_{\mathrm{m}}\boldsymbol{\varepsilon}_{\mathrm{m}}$$

$$\bar{\boldsymbol{\sigma}} = f_{\Omega}\boldsymbol{\sigma}_{\Omega} + f_{\mathrm{m}}\boldsymbol{\sigma}_{\mathrm{m}}$$

$$\boldsymbol{\sigma}_{\Omega} = \mathbf{D}_{\Omega} : \boldsymbol{\varepsilon}_{\Omega}$$

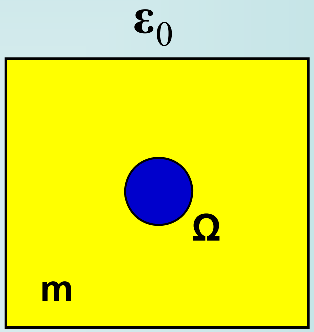
$$\boldsymbol{\sigma}_{\mathrm{m}} = \mathbf{D}_{\mathrm{m}} : \boldsymbol{\varepsilon}_{\mathrm{m}}$$

Eshelby matrix-inclusion solution

$$\boldsymbol{\varepsilon}_{\mathrm{c}} = \mathbf{S}_{\Omega} : \boldsymbol{\varepsilon}_{\mathrm{t}}$$

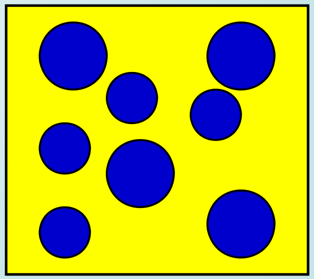
$$\boldsymbol{\varepsilon}_{\Omega} = \left(\mathbf{I}^{4s} + \mathbf{S}_{\Omega} \cdot \mathbf{A}_{\Omega}\right) : \boldsymbol{\varepsilon}_{0}$$

Where $\mathbf{A}_{\Omega} = \left[\left(\mathbf{D}_{\Omega} - \mathbf{D}_{\mathrm{m}}\right)\cdot \mathbf{S}_{\Omega} + \mathbf{D}_{\mathrm{m}}\right]^{-1} \cdot \left(\mathbf{D}_{\mathrm{m}} - \mathbf{D}_{\Omega}\right)$



Mori-Tanaka averaging

$$\boldsymbol{\varepsilon}_{\Omega} = \left(\mathbf{I}^{4s} + \mathbf{S}_{\Omega} \cdot \mathbf{A}_{\Omega}\right) : \boldsymbol{\varepsilon}_{\mathrm{m}}$$



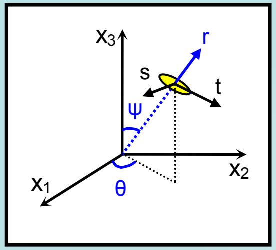
Two-phase composite

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}_{\mathrm{m}\Omega} : \bar{\boldsymbol{\varepsilon}}$$

Where $\mathbf{D}_{\mathrm{m}\Omega} = \left(f_{\Omega}\mathbf{D}_{\Omega} \cdot \mathbf{T}_{\Omega} + f_{\mathrm{m}}\mathbf{D}_{\mathrm{m}}\right) \cdot \left(f_{\Omega}\mathbf{T}_{\Omega} + f_{\mathrm{m}}\mathbf{I}^{4s}\right)^{-1}$

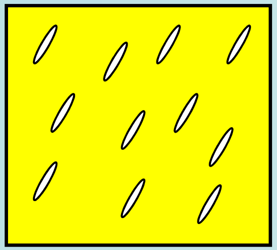
$$\mathbf{T}_{\Omega} = \mathbf{I}^{4s} + \mathbf{S}_{\Omega} \cdot \mathbf{A}_{\Omega}$$

Microcracking



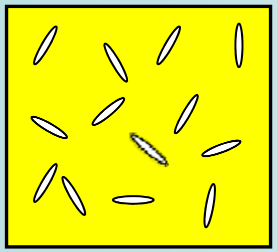
$$\overline{\boldsymbol{\varepsilon}} = f_{\Omega}\boldsymbol{\varepsilon}_{\Omega} + f_m \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_a$$

Added strains from a series of cracks with the same orientation



$$\boldsymbol{\varepsilon}_{\alpha} = f\,\frac{16(1-\upsilon_m{}^2)}{3E_m}\begin{bmatrix} \sigma_{rr} \\ \dfrac{4}{2-\upsilon_m}\sigma_{rs} \\ \dfrac{4}{2-\upsilon_m}\sigma_{rt} \end{bmatrix} = \frac{\omega}{1-\omega}\cdot\frac{1}{E_m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{4}{2-\upsilon_m} & 0 \\ 0 & 0 & \dfrac{4}{2-\upsilon_m} \end{bmatrix} : \begin{bmatrix} \sigma_{rr} \\ \sigma_{rs} \\ \sigma_{rt} \end{bmatrix} = \mathbf{C}_a(\omega) : \boldsymbol{\sigma}_L$$

Added strains from a series of cracks with multiple orientations



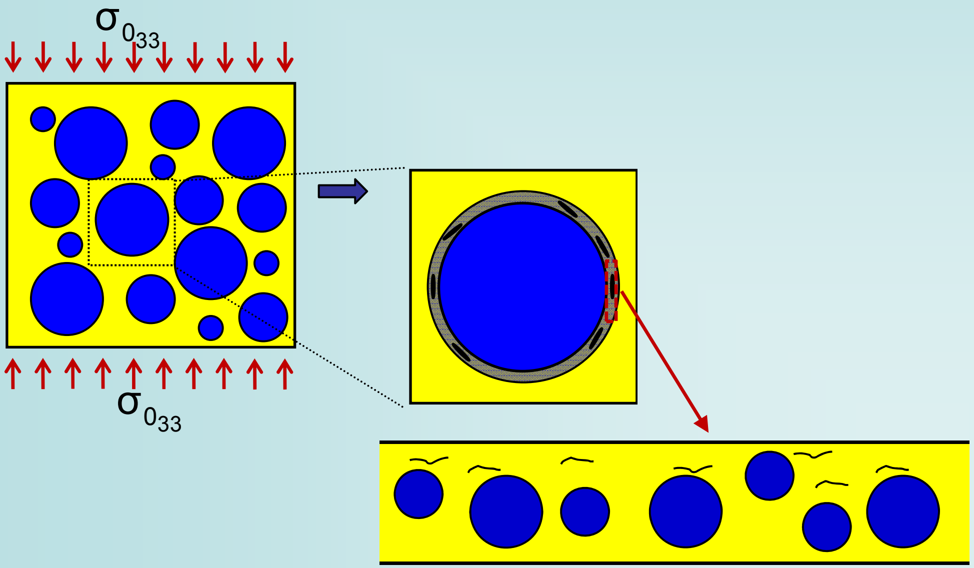
$$\boldsymbol{\varepsilon}_a = \left(\frac{1}{2\pi} \int_{2\pi} \int_{\pi/2} \mathbf{N}_{\varepsilon} : \mathbf{C}_a(\omega) : \mathbf{N} \sin(\psi)\, d\psi\, d\theta \right) : \overline{\boldsymbol{\sigma}}$$

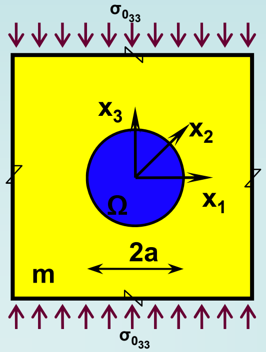
Average stress – average strain relationship:

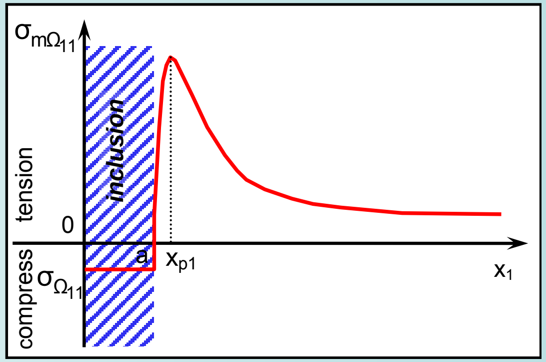
$$\overline{\boldsymbol{\sigma}} = \left(\mathbf{I}^{4s} + \frac{\mathbf{D}_{m\Omega}}{2\pi} \cdot \int_{2\pi} \int_{\pi/2} \mathbf{N}_{\varepsilon} \cdot \mathbf{C}_a(\omega) \cdot \mathbf{N} \sin(\psi)\, d\psi\, d\theta \right)^{-1} \cdot \mathbf{D}_{m\Omega} : \overline{\boldsymbol{\varepsilon}}$$

Crack initiation

Exterior point Eshelby solution used for a crack initiation criterion







$$\boldsymbol{\varepsilon}_{m\Omega}(\mathbf{x}) = \mathbf{T}(\mathbf{x}) : [\, f_{\Omega}\mathbf{T}_{\Omega} + f_{m}\mathbf{I}^{4s}]^{-1} : \bar{\boldsymbol{\varepsilon}}$$

$$\mathbf{T}(\mathbf{x}) = \mathbf{I}^{4s} - \mathbf{S}^{E}(\mathbf{x}) \cdot [\mathbf{S}_{\Omega} + (\mathbf{D}_{\Omega} - \mathbf{D}_{m})^{-1} \cdot \mathbf{D}_{m}]^{-1}$$

$$s_{m\mathbf{I}} - f_{s} = 0 \quad \rightarrow \quad \zeta(\mathbf{e}_{\omega}) = e_{\omega rr} \cdot \frac{1+\alpha_L}{2} + \sqrt{\left(\frac{1-\alpha_L}{2}\right)^2 e_{\omega rr}{}^2 + r_{\zeta}{}^2 \cdot \gamma^2}$$

$$\gamma = \sqrt{e_{\omega rs}{}^2 + e_{\omega rt}{}^2}$$

$$\alpha_L = \upsilon_m / (1-\upsilon_m)$$

$$r_{\zeta} = \frac{\upsilon_m - 1/2}{\upsilon_m - 1}$$

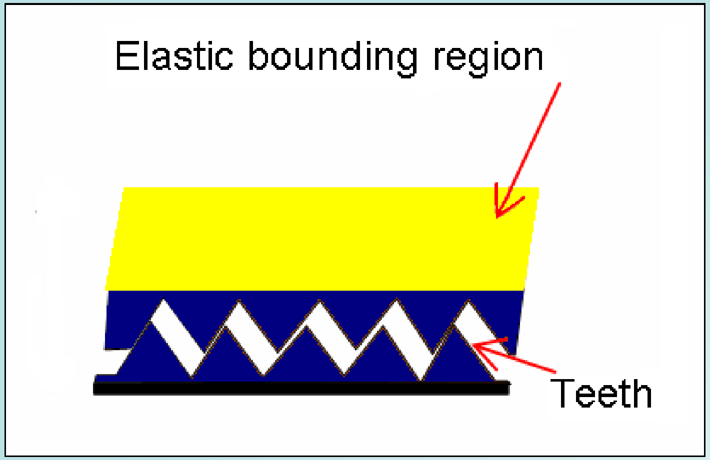
$$\mathbf{e}_{\omega} = (1-\omega)\mathbf{C}_{Lm} : \mathbf{N} : \mathbf{D}_{m} \cdot \mathbf{T}(\mathbf{x}_{p})\left(f_{\Omega}\mathbf{T}_{\Omega} + f_{m}\mathbf{I}^{4s}\right)^{-1} : \bar{\boldsymbol{\varepsilon}} + \omega \mathbf{N}_{\varepsilon} : \bar{\boldsymbol{\varepsilon}}$$

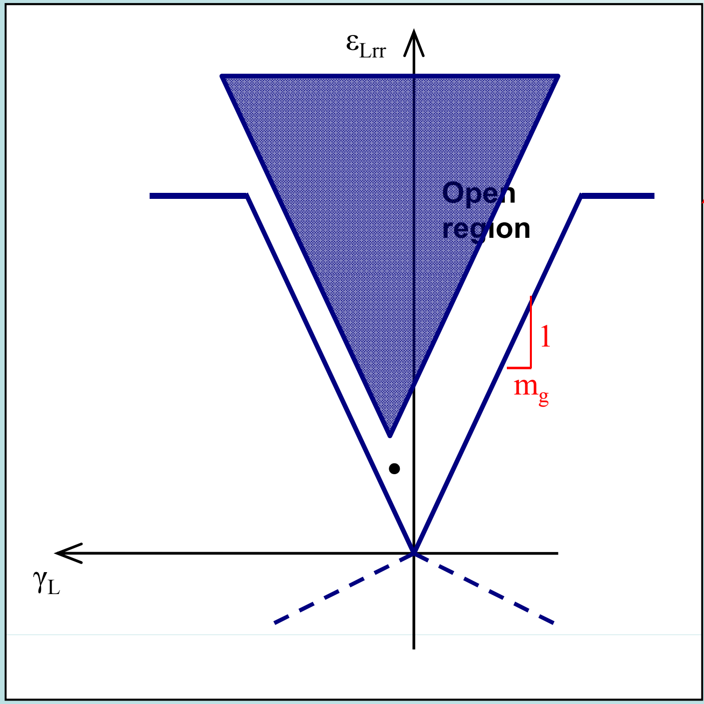
elastic strain in matrix (EPE) — local average strain

$$S^{E}_{ijmn}(\mathbf{x}) = \frac{\rho^3}{30(1-\upsilon_m)}[(3\rho^3 + 10\upsilon_m - 5)\delta_{ij}\delta_{mn} + (3\rho^3 - 10\upsilon_m + 5)\cdot(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + 15(1-\rho^2)\delta_{ij}\bar{x}_m\bar{x}_n +$$
$$+15(1-2\upsilon_m - \rho^2)\delta_{mn}\bar{x}_i\bar{x}_j + 15(\upsilon_m - \rho^2)\cdot(\delta_{im}\bar{x}_j\bar{x}_n + \delta_{in}\bar{x}_j\bar{x}_m + \delta_{jm}\bar{x}_i\bar{x}_n + \delta_{jn}\bar{x}_i\bar{x}_m) + 15(7\rho^2 - 5)\bar{x}_i\bar{x}_j\bar{x}_m\bar{x}_n]$$

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad \rho = \frac{a}{|\mathbf{x}|}$$

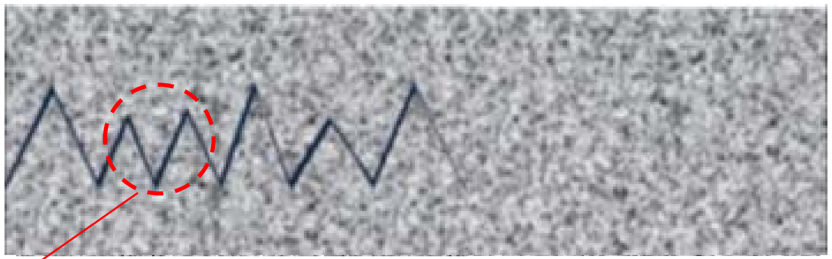
Rough crack contact





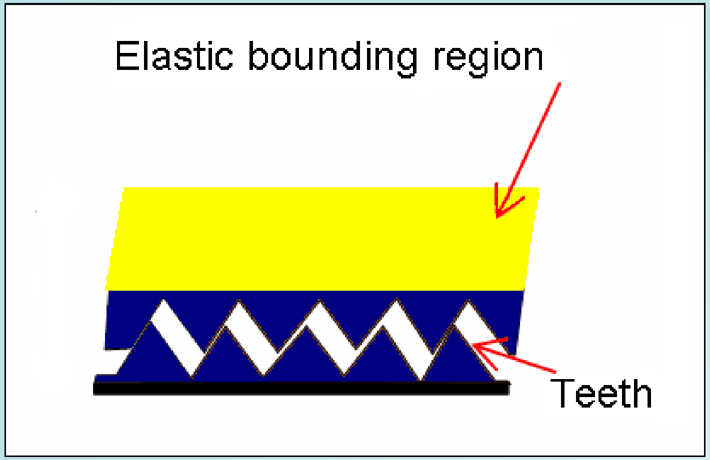
Damaged **Undamaged**

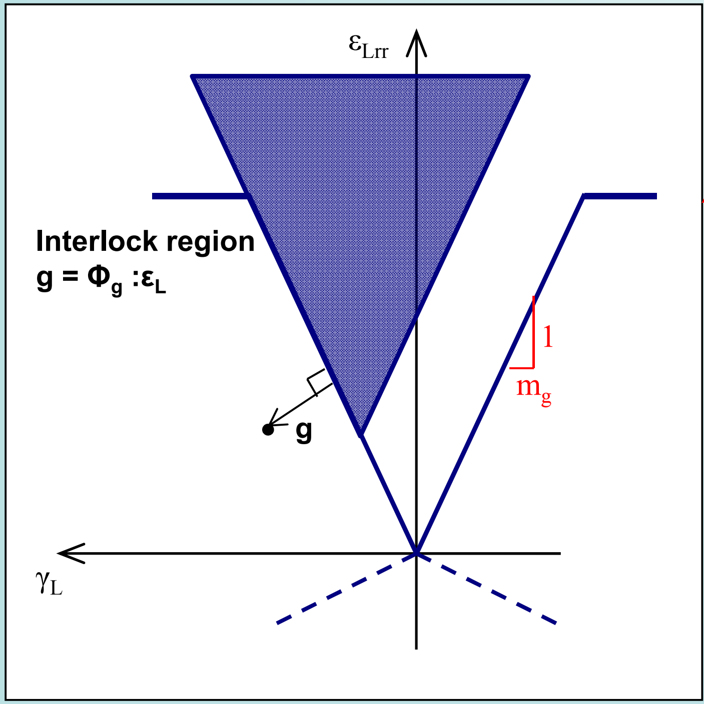
$$\boldsymbol{\sigma}_L = \omega\, H_f \mathbf{D}_L \boldsymbol{\Phi} \boldsymbol{\varepsilon}_L + (1-\omega)\mathbf{D}_L \boldsymbol{\varepsilon}_L$$



$\boldsymbol{\Phi} = \mathbf{0}$ **Crack open**

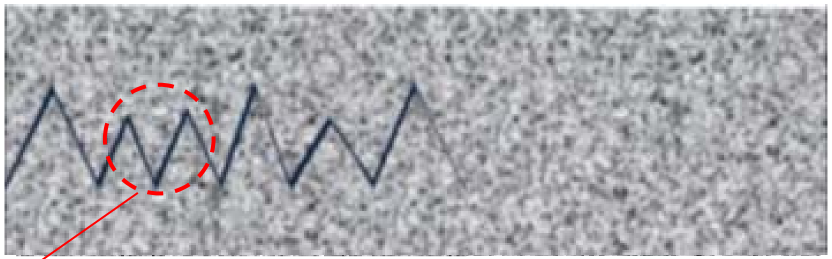
Rough crack contact





Damaged **Undamaged**

$$\boldsymbol{\sigma}_L = \omega \, H_f \mathbf{D}_L \boldsymbol{\Phi} \boldsymbol{\varepsilon}_L + (1-\omega) \mathbf{D}_L \boldsymbol{\varepsilon}_L$$

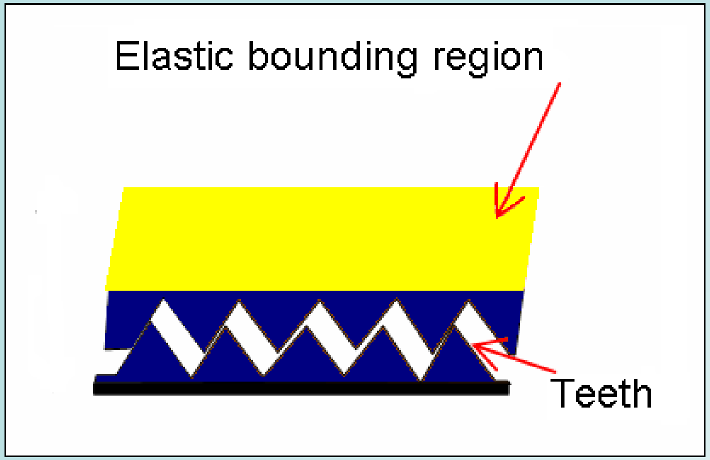


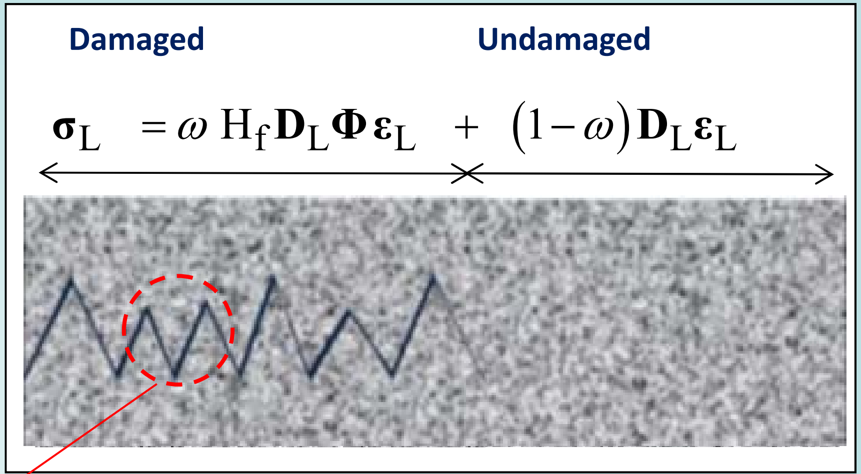
$\boldsymbol{\Phi} = \mathbf{0}$ Crack open

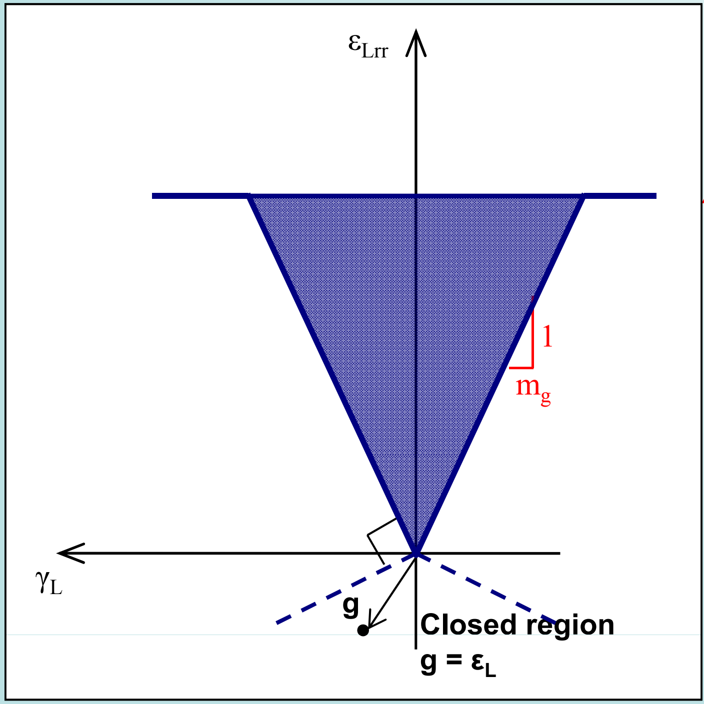
$\boldsymbol{\Phi} = \boldsymbol{\Phi}_g$ **Shear contact**

$$\boldsymbol{\Phi}_{\mathbf{g}} = \frac{1}{1+m_g^{\,2}} \begin{bmatrix} m_g^{\,2} & -m_g \\ -m_g & 1 \end{bmatrix}$$

Rough crack contact







$\Phi = 0$ Crack open

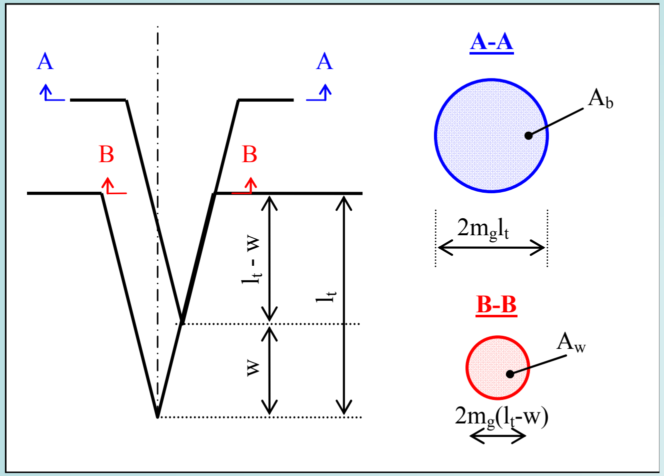
$\Phi = \Phi_g$ Shear contact

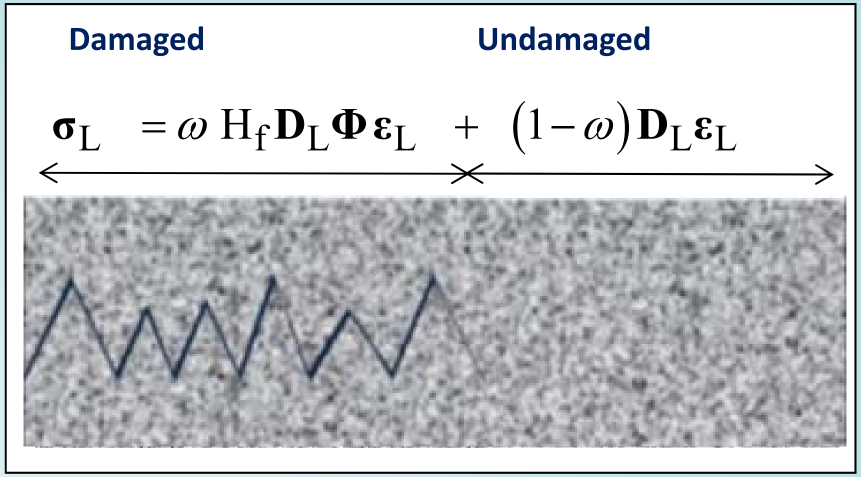
$\Phi = I$ **Crack closed**

$$\Phi_g = \frac{1}{1+m_g^2}\begin{bmatrix} m_g^2 & -m_g \\ -m_g & 1 \end{bmatrix}$$

Rough crack contact

H_f - Shear contact reduction function





$$H_f = \left(1 - \frac{\varepsilon_{Lrr} - \varepsilon_{tm}}{\varepsilon_u}\right)^2 \rightarrow \boxed{H_f = e^{-2 \cdot \frac{\varepsilon_{Lrr} - \varepsilon_{tm}}{\varepsilon_u}}}$$

Include contact in local compliance tensor

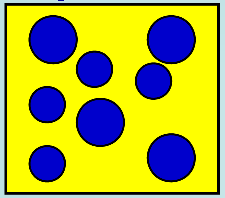
$$\mathbf{C_a} = \left[\left[(1-\omega)\mathbf{I}^{4s} + \omega \sum_i p_i H_{fi} \boldsymbol{\Phi}_i\right]^{-1} - \mathbf{I}^{4s}\right] : \mathbf{C}_{Lm\Omega}$$

Final stress-strain model

$$\bar{\boldsymbol{\sigma}} = \left(\mathbf{I}^{4s} + \mathbf{D}_{m\Omega} \cdot \frac{1}{2\pi} \int_{2\pi} \int_{\pi/2} \mathbf{N}_\varepsilon \cdot \mathbf{C_a} \cdot \mathbf{N} \sin(\psi)\, d\psi\, d\theta\right)^{-1} \cdot \mathbf{D}_{m\Omega} : \bar{\boldsymbol{\varepsilon}}$$

Model components

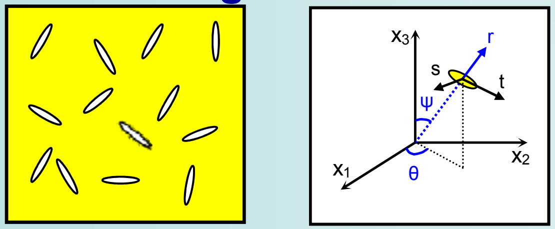
1. Two-phase composite - Eshelby solution and Mori-Tanaka averaging



$$\bar{\boldsymbol{\sigma}} = \mathbf{D}_{\mathrm{m}\Omega} : \left(\bar{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_{\mathrm{a}}\right)$$

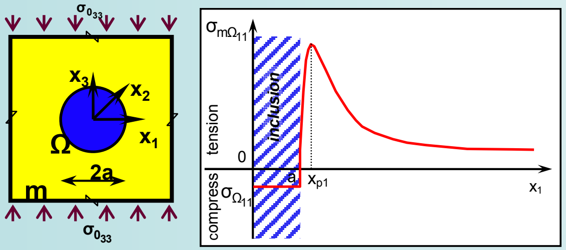
$$\mathbf{D}_{m\Omega} = \left(\mathrm{f}_{\Omega}\mathbf{D}_{\Omega} \cdot \mathbf{T}_{\Omega} + \mathrm{f}_{\mathrm{m}}\mathbf{D}_{\mathrm{m}}\right) \cdot \left(\mathrm{f}_{\Omega}\mathbf{T}_{\Omega} + \mathrm{f}_{\mathrm{m}}\mathbf{I}^{4s}\right)^{-1}$$

2. Microcracking - Nemat–Nasser and Hori, Budiansky and O'Connell



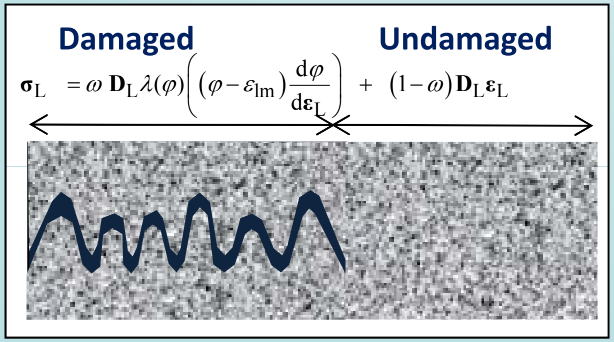
$$\boldsymbol{\varepsilon}_{\mathrm{a}} = \left(\frac{1}{2\pi}\int_{2\pi}\int_{\pi/2}\mathbf{N}_{\varepsilon} \cdot \frac{\omega}{1-\omega}\mathbf{C}_{\mathrm{L}} \cdot \mathbf{N}\sin(\psi)\,d\psi\,d\theta\right) \cdot \bar{\boldsymbol{\sigma}}$$

3. Crack initiation – Exterior point Eshelby solution



$$\mathbf{e}_{\omega} = (1-\omega)\mathbf{C}_{\mathrm{Lm}} \cdot \mathbf{N} \cdot \mathbf{D}_{\mathrm{m}} \cdot \mathbf{T}(\mathbf{x}_{p})\left(\mathrm{f}_{\Omega}\mathbf{T}_{\Omega} + \mathrm{f}_{\mathrm{m}}\mathbf{I}^{4s}\right)^{-1} : \bar{\boldsymbol{\varepsilon}} + \omega\mathbf{N}_{\varepsilon} : \bar{\boldsymbol{\varepsilon}}$$

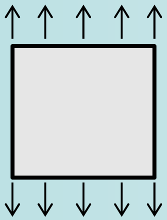
4. Smoothed rough contact

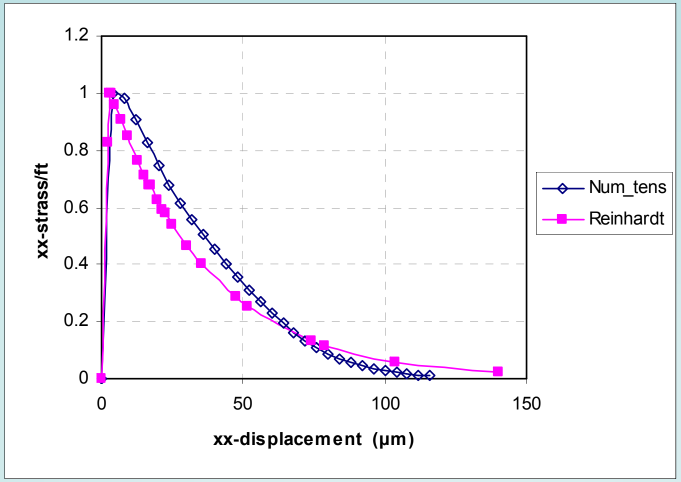


$$\lambda(\varphi) = \frac{1}{2}\left[1 - \frac{\varphi}{\varphi-\varepsilon_{\mathrm{lm}}}\ln\left(\frac{\cosh\left(\varphi/\varepsilon_{\mathrm{Z}}\right)}{\cosh\left(\varepsilon_{\mathrm{lm}}\right)}\right)\right]$$

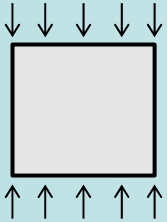
Numerical results

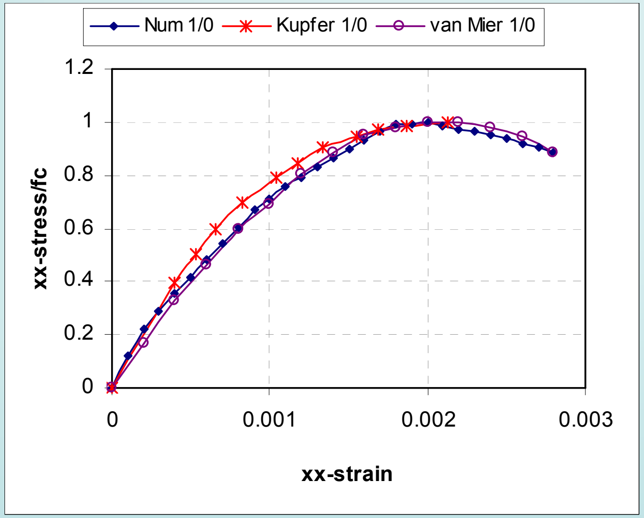
Uniaxial tension

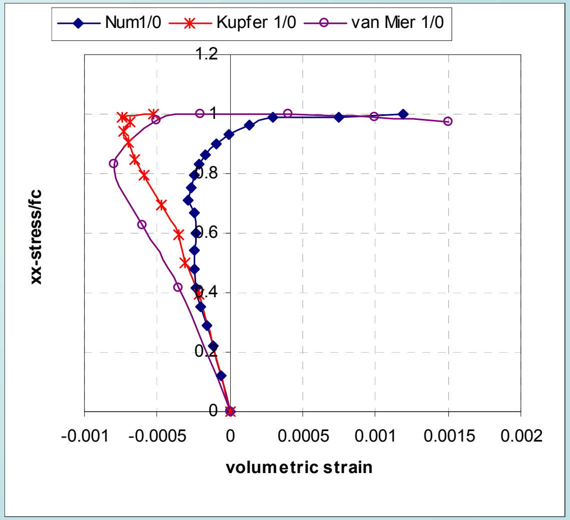




Uniaxial compression

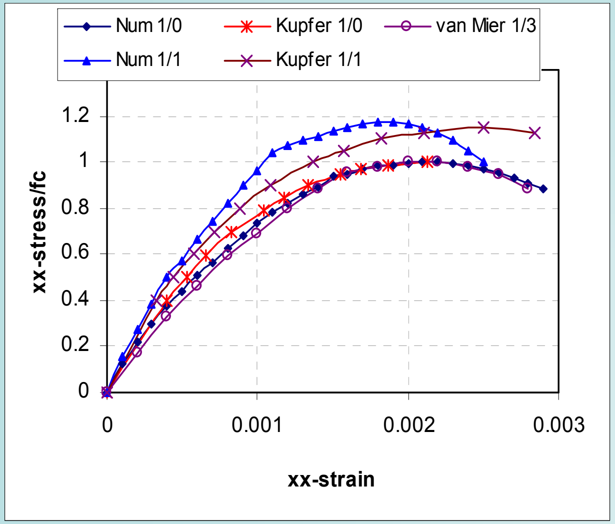


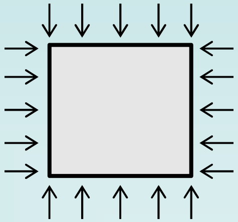




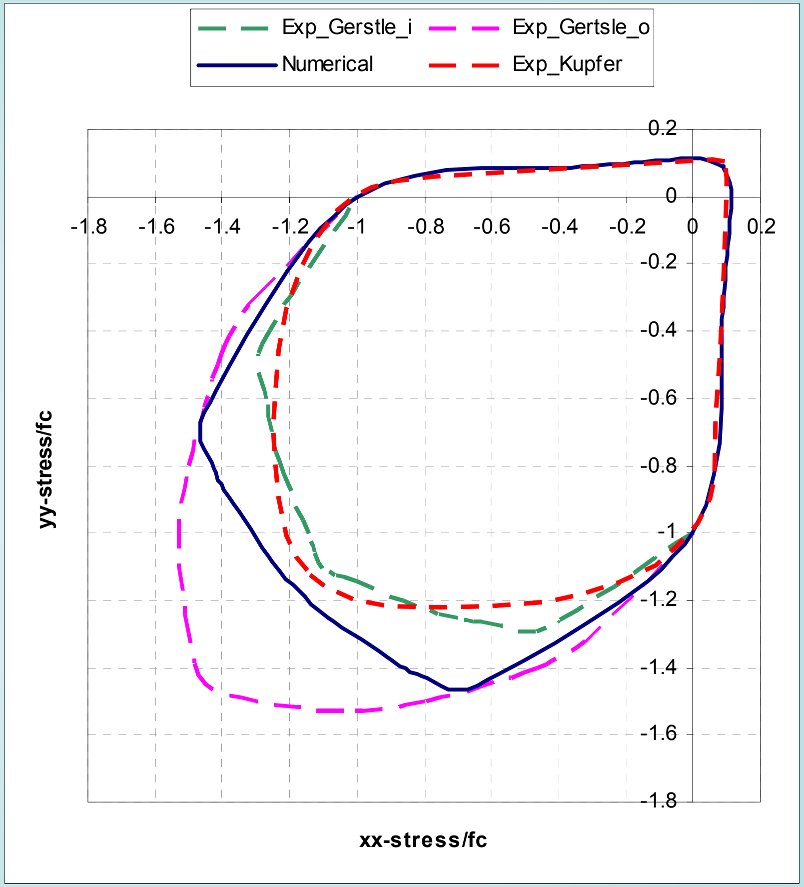
Numerical results

Biaxial compression 1/1

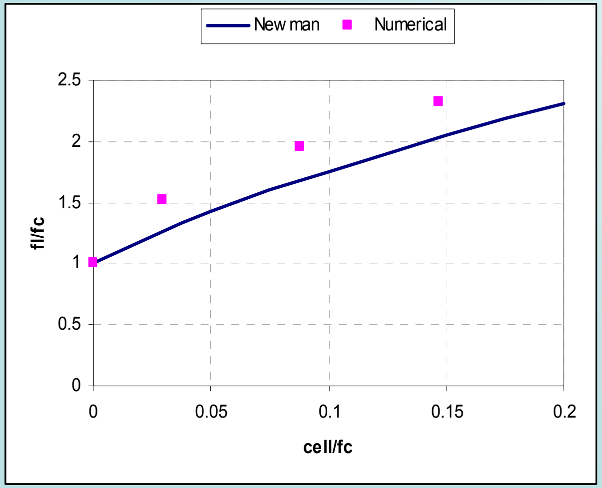


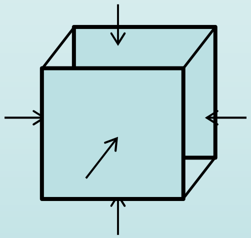


Biaxial envelope



Triaxial strengths





Concluding remarks

By simulating specific physical mechanisms at micro and meso scale; e.g. matrix – spherical inclusion composite, microcrack initiation and propagation, and stress recovery through rough crack closure; the proposed micro-mechanical model captures fundamental characteristics of the overall macroscopic behaviour: damage induced anisotropy, dilatancy, realistic biaxial failure envelope and reasonable triaxial at relatively low confining stresses.

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Direct tensor notation (*Voyiadjis and Kattan, 2006)*

Table A.1. Direct tensor notation

Direct tensor notation	Notation based on Einstein summation convention (summation of repeated indices)
$\alpha = \mathbf{a} \cdot \mathbf{b}$	$\alpha = \mathrm{a}_i \mathrm{b}_i$
$\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$	$\mathrm{A}_{ij} = \mathrm{a}_i \mathrm{b}_j$
$\alpha = \mathbf{A} : \mathbf{B}$	$\alpha = \mathrm{A}_{ij} \mathrm{B}_{ij}$
$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$	$\mathrm{C}_{ik} = \mathrm{A}_{ij} \mathrm{B}_{jk}$
$\mathbf{P} = \mathbf{A} \otimes \mathbf{B}$	$\mathrm{P}_{ijkl} = \mathrm{A}_{ij} \mathrm{B}_{kl}$
$\mathbf{C} = \mathbf{P} : \mathbf{A}$	$\mathrm{C}_{ij} = \mathrm{P}_{ijkl} \mathrm{A}_{kl}$
$\mathbf{B} = \mathbf{A} : \mathbf{P}$	$\mathrm{B}_{kl} = \mathrm{A}_{ij} \mathrm{P}_{ijkl}$
$\mathbf{R} = \mathbf{P} \cdot \mathbf{Q}$	$\mathrm{R}_{ijmn} = \mathrm{P}_{ijkl} \mathrm{Q}_{klmn}$

Direct tensor and matrix representations

Stresses are directly equivalent in matrix and direct tensor notation whereas strains are different, as shown below.

$$\boldsymbol{\sigma}_{matrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \boldsymbol{\sigma}_{dir\_tensor} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_{matrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \quad \text{whereas } \boldsymbol{\varepsilon}_{dir\_tensor} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}$$