ACME School

4th April 2011

The Boundary Element Method in Elastostatics

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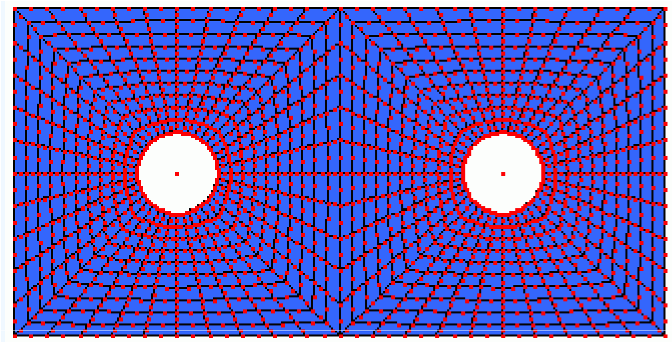
Overview

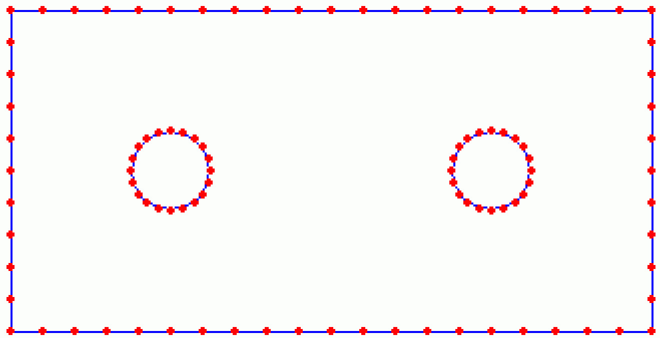
- Introduction
- Reciprocal theorem
- Fundamental solutions
- Boundary integral equation
- Boundary element method
- Re-analysis and interactivity
- Enrichment of approximation space

Introduction – a cursory overview



FEM

More versatile
Domain method



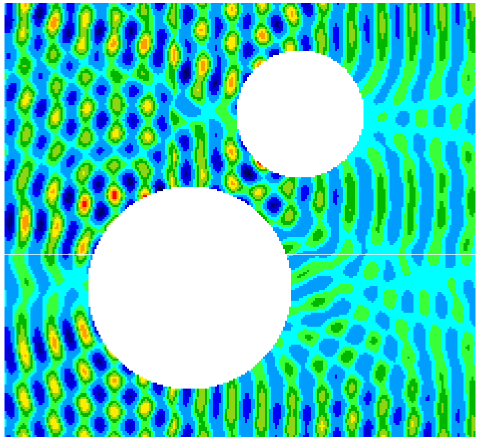
BEM

Computationally more demanding

Simple meshing

Solution accuracy

Infinite domain problems



Elements are only used on the two internal boundaries in this wave problem

Introduction – prelude to the theoretical development

For the purposes of this lecture we will start by stating the reciprocal theorem from a structural mechanics viewpoint.

The notes contain a fuller description with greater mathematical rigour. The reciprocal theorem is developed from a weighted residual expression.

We will confine ourselves to the collocation BEM. There is a popular Galerkin form of BEM too.

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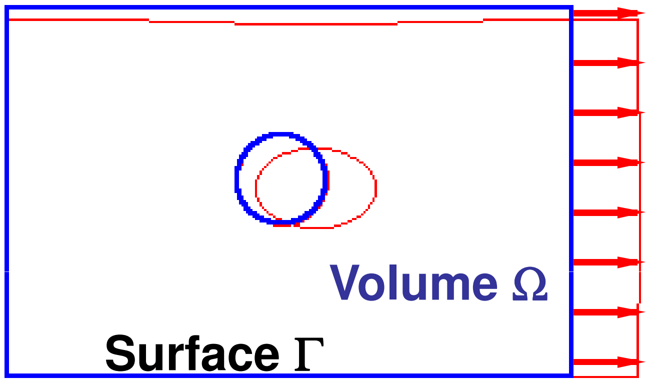
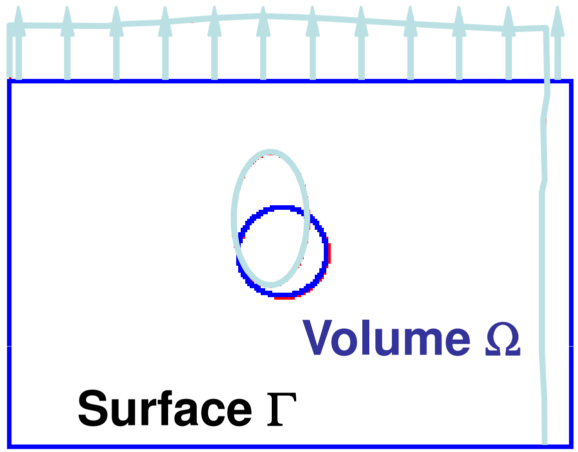
Reciprocal theorem





Real load case:

Tractions: t_i

Displacements: u_i

Body forces: b_i

Complementary load case:

Tractions: t_i^*

Displacements: u_i^*

Body forces: b_i^*

Reciprocal theorem

Form statements of work done by **force** x **displacement**
force x **displacement**

$$\int_\Gamma t_i^* u_i \, d\Gamma + \int_\Omega b_i^* u_i \, d\Omega = \int_\Gamma u_i^* t_i \, d\Gamma + \int_\Omega u_i^* b_i \, d\Omega$$

Integrate surface tractions over the surface Γ

Integrate body forces over the volume Ω

This is the ***reciprocal theorem*** due to Maxwell and Betti.

Fundamental solutions

Reciprocal theorem statement:

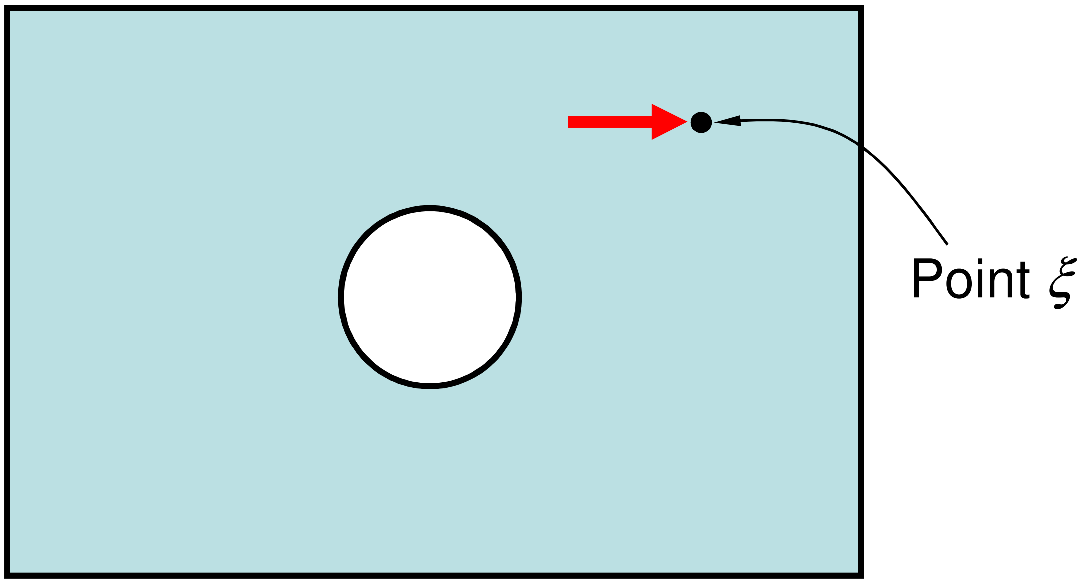
$$\int_\Gamma t_i^* u_i d\Gamma + \int_\Omega b_i^* u_i d\Omega = \int_\Gamma u_i^* t_i d\Gamma + \int_\Omega u_i^* b_i d\Omega$$

In order to reduce to boundary-only, we need to eliminate the two volume integrals.

- We will simplify the development for this lecture by stipulating no body forces in the real load case, i.e. $b_i = 0$
- We will address the volume integral on the LHS by appropriate choice of a complementary load case

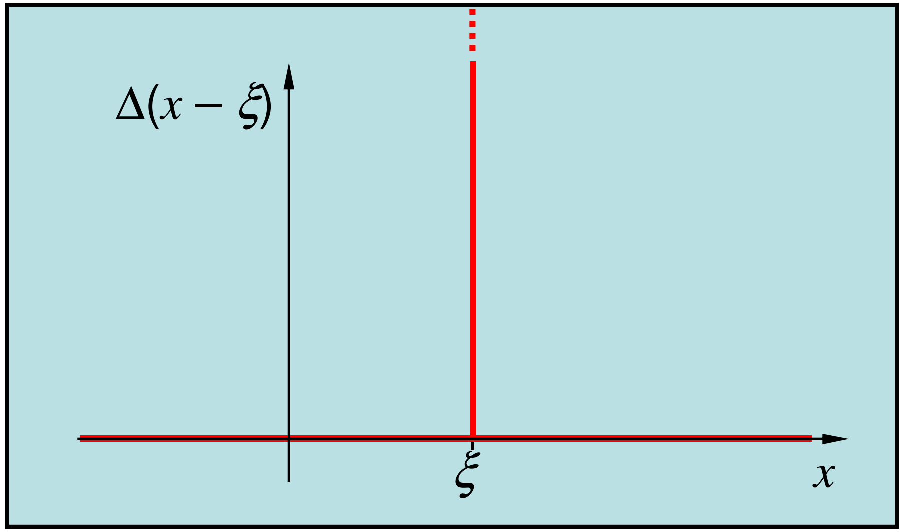
Fundamental solutions

Complementary load case *: Dirac delta function point load in one of the coordinate directions at some point ξ



Fundamental solutions

Properties of Dirac delta function:



$$\Delta(x-\xi) = \begin{cases} \infty & x = \xi \\ 0 & x \neq \xi \end{cases}$$

$$\int_a^b \Delta(x-\xi)\,dx = 1, \qquad a < \xi < b$$

$$\int_a^b g(x)\Delta(x-\xi)\,dx = g(\xi), \qquad a < \xi < b$$

Fundamental solutions

So the displacement field in the complementary load case, u^*, is the solution to the equilibrium equation:

$$\sigma_{ij,j} + \Delta\left(x - \xi\right) e_i(x) = 0$$

Fundamental solutions

So the displacement field in the complementary load case, u^*, is the solution to the equilibrium equation:

$$\sigma_{ij,j} + \Delta\left(x - \xi\right) e_i(x) = 0$$

It turns out that the solution is:

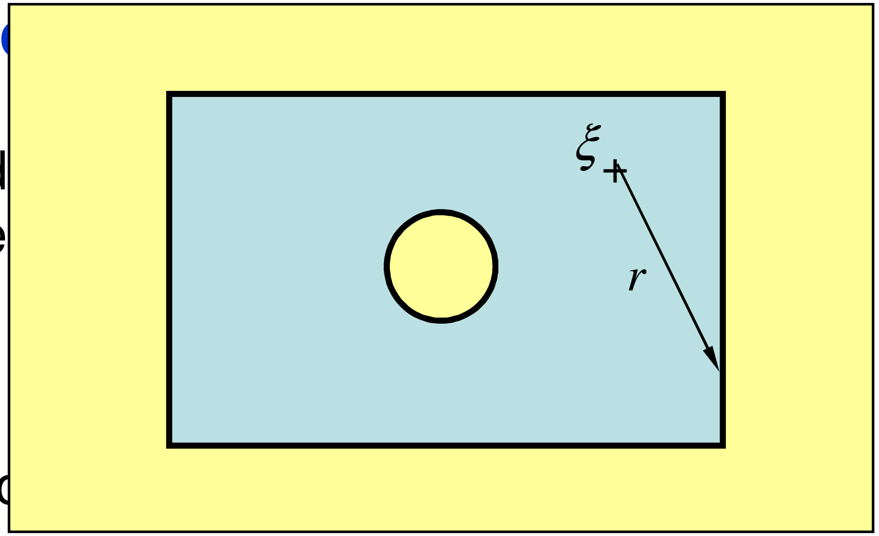
$$u_i^* = U_{ik} e_k$$

where

$$U_{ik} = \frac{1}{8\pi\mu\left(1-\nu\right)}\left[\left(3-4\nu\right)\ln\left(\frac{1}{r}\right)\delta_{ik} + r_{,i}r_{,k}\right] \qquad \text{(2D)}$$

$$U_{ik} = \frac{1}{16\pi\mu\left(1-\nu\right)r}\left[(3-4\nu)\delta_{ik} + r_{,i}r_{,k}\right] \qquad \text{(3D)}$$

Fundamental soluti



So the displacement field

u^*, is the solution to the e

$$\sigma_{ij,j} + \Delta$$

It turns out that the soluti

$$u_i^* = U_{ik} e_k$$

where

$$U_{ik} = \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu) \ln\left(\frac{1}{r}\right) \delta_{ik} + r_{,i} r_{,k} \right] \qquad \text{(2D)}$$

$$U_{ik} = \frac{1}{16\pi\mu(1-\nu) r} \left[(3-4\nu)\delta_{ik} + r_{,i} r_{,k} \right] \qquad \text{(3D)}$$

Fundamental solutions

Displacement fundamental solutions:

$$U_{ik} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ik} + r_{,i}r_{,k}\right] \qquad \text{(2D)}$$

$$U_{ik} = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ik} + r_{,i}r_{,k}\right] \qquad \text{(3D)}$$

Differentiate and apply Hooke's Law to arrive at the corresponding traction fundamental solutions:

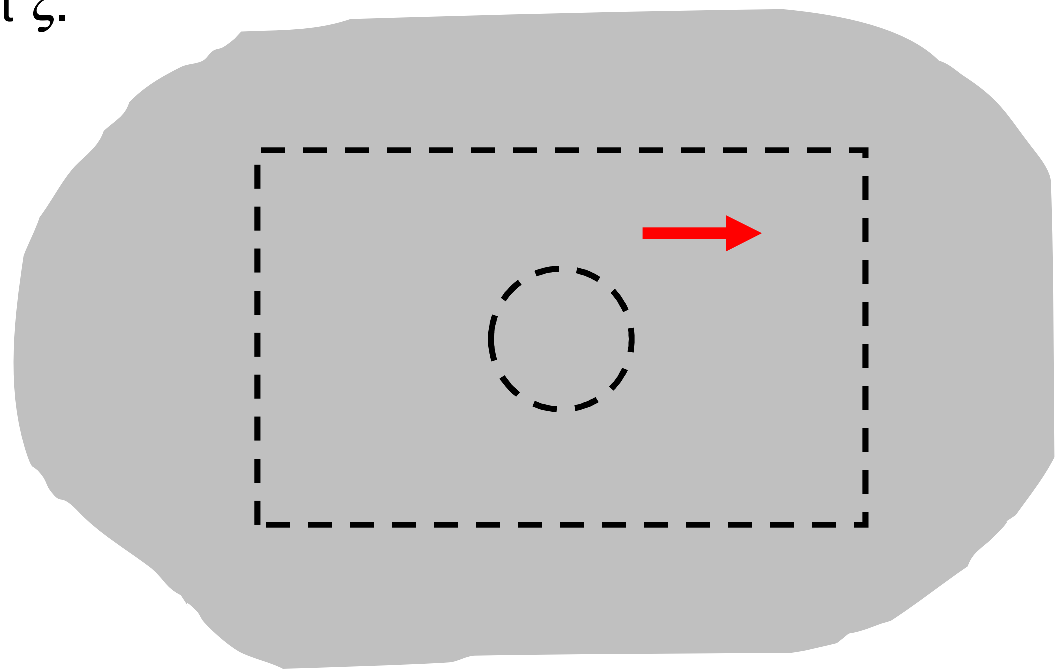
$$T_{ik} = \frac{-1}{4\pi(1-\nu)r}r_{,n}\left[(1-2\nu)\delta_{ik} + 2r_{,i}r_{,k}\right] + \frac{1-2\nu}{4\pi(1-\nu)r}\left(r_{,k}n_i - r_{,i}n_k\right) \qquad \text{(2D)}$$

$$T_{ik} = \frac{-1}{8\pi(1-\nu)r^2}r_{,n}\left[(1-2\nu)\delta_{ik} + 3r_{,i}r_{,k}\right] + \frac{1-2\nu}{8\pi(1-\nu)r^2}\left(r_{,k}n_i - r_{,i}n_k\right) \qquad \text{(3D)}$$

Fundamental solutions

Physical significance:

Fundamental solutions provide the displacement and traction fields, in an infinite material, due to a Dirac point force at ξ.



These solutions are due to Kelvin.

Boundary integral equation

Use of the Dirac delta function as the complementary load case has given us a set of fundamental solutions from which we can easily find u^* and t^*.

The choice of the Dirac delta function also removes the remaining volume integral in the reciprocal theorem statement:

0 if $b_i = 0$

$$\int_\Gamma t_i^* u_i d\Gamma + \int_\Omega b_i^* u_i d\Omega = \int_\Gamma u_i^* t_i d\Gamma + \int_\Omega u_i^* b_i d\Omega$$

$$\int_\Omega b_i^* u_i d\Omega = \int_\Omega \Delta\left(x-\xi\right) e_i u_i d\Omega = u_i\left(\xi\right) e_i$$

Boundary integral equation

Treatment of the two volume integrals leaves simply:

$$u_k(\xi) + \int_\Gamma T_{ik} u_i \, d\Gamma = \int_\Gamma U_{ik} t_i \, d\Gamma \qquad (*)$$

The last step in making this a boundary-only expression is to ***move ξ to the boundary***, i.e. $\xi \in \Gamma$.

This causes complications because, when $\xi \in \Gamma$, r passes through zero on the boundary causing both boundary integrals to ***contain singular functions***.

Boundary integral equation

The integrals containing U_{ik} are only weakly singular. The integrals in 2D, for example, have a logarithmic singularity and can be quickly evaluated using the logarithmic form of Gauss-Legendre quadrature.

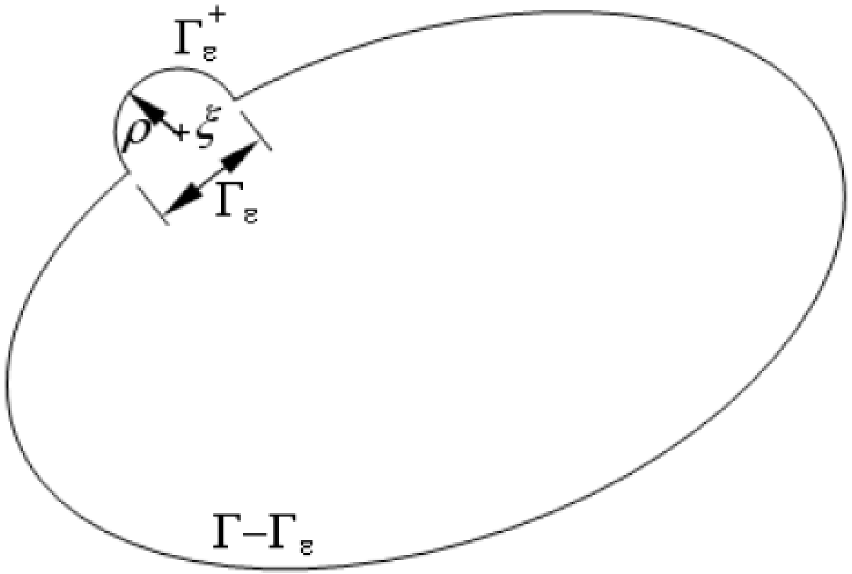
$$\int_{-1}^{1} \ln\left(\frac{1}{x}\right) f\left(x\right) dx \simeq \sum_{i=1}^{N} f\left(x_i\right) w_i$$

There are other schemes – mostly involving coordinate transformation – for evaluating weakly singular integrals.

Boundary integral equation

The strongly singular integrals containing T_{ik} may be taken in the *Cauchy principal value sense* (limit as radius $\rho \rightarrow 0$) causing the introduction of a multiplier c_{ik}.

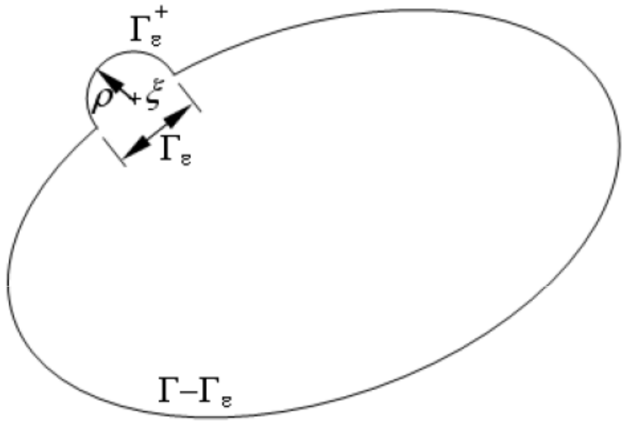
$$c_{ik}(\xi)\, u_k(\xi) + \int_{\Gamma} T_{ik} u_i \, d\Gamma = \int_{\Gamma} U_{ik} t_i \, d\Gamma$$



We will now see how this arises…

Boundary integral equation

These integrals may be split into three parts:



$$\int_\Gamma T_{ik} u_i d\Gamma = \lim_{\rho\to 0}\left\{\int_{\Gamma-\Gamma_\varepsilon} T_{ik} u_i d\Gamma\right\}$$

$$+ \lim_{\rho\to 0}\left\{\int_{\Gamma_\varepsilon^+} T_{ik}\left(u_i(x) - u_i(\xi)\right) d\Gamma(x)\right\} \longrightarrow 0$$

$$+ u_i(\xi)\lim_{\rho\to 0}\left\{\int_{\Gamma_\varepsilon^+} T_{ik} d\Gamma\right\}$$

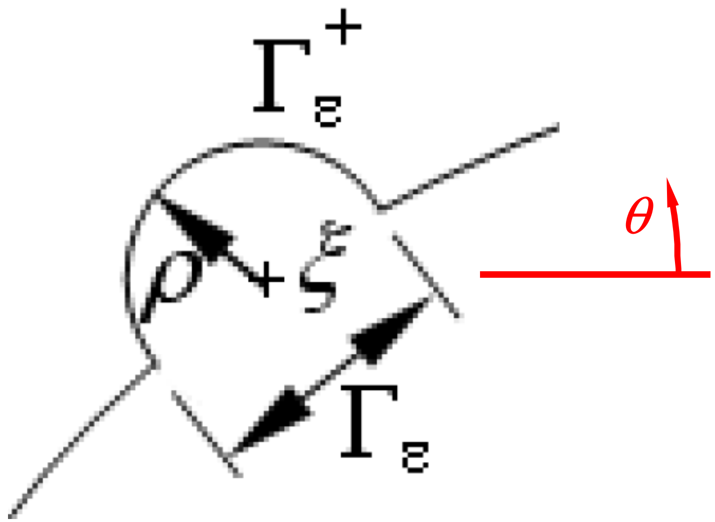
Boundary integral equation

For convenience we write the last term as:

$$u_i(\xi) \lim_{\rho \to 0} \left\{ \int_{\Gamma_\varepsilon^+} T_{ik} d\Gamma \right\} = \alpha_{ik}(\xi)\, u_i(\xi)$$

So $c_{ik} = \delta_{ik} + \alpha_{ik}$

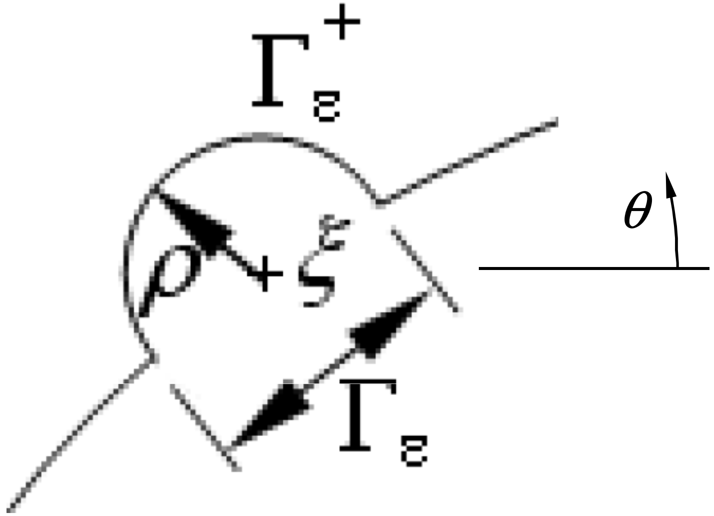
Also define θ coordinate:



Boundary integral equation

On Γ_ε^+:

$$r = \rho \cos\theta e_1 + \rho \sin\theta e_2$$
$$d\Gamma_\varepsilon^+ = \rho d\theta$$
$$r_{,n} = 1$$
$$r_{,1} = \cos\theta$$
$$r_{,2} = \sin\theta$$
$$n_1 = \cos\theta$$
$$n_2 = \sin\theta$$



Substituting these into the various T_{ik} terms allows the integrals to be calculated analytically, yielding α_{ik} and then c_{ik}

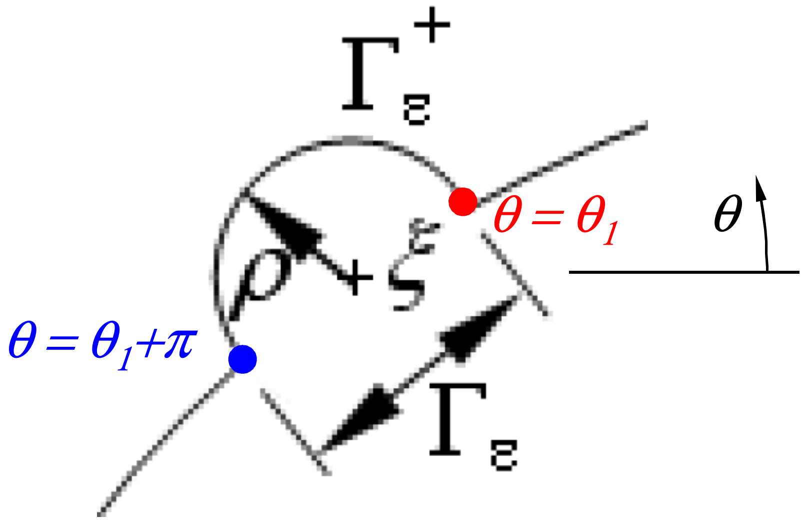
Boundary integral equation

Worked example: α_{11} on a smooth boundary

Substituting the functions of r and n into T_{11} gives

$$\alpha_{11}(\xi) = \int_{\Gamma_\varepsilon^+} \left[\frac{-1}{4\pi(1-\nu)\rho} \left(1 - 2\nu + 2\cos^2\theta\right) \right] d\Gamma_\varepsilon^+$$

$$\alpha_{11}(\xi) = \frac{-1}{4\pi(1-\nu)} \int_{\theta_1}^{\theta_1+\pi} \left(1 - 2\nu + 2\cos^2\theta\right) d\theta$$



Boundary integral equation

Worked example: α_{11} on a smooth boundary

$$\alpha_{11}(\xi) = \frac{-1}{4\pi(1-\nu)} \left[(1-2\nu)\theta + (\theta + \cos\theta\sin\theta)\right]_{\theta_1}^{\theta_1+\pi}$$

$$\alpha_{11}(\xi) = \frac{-1}{4\pi(1-\nu)} \left\{(2-2\nu)\pi + \cos(\theta_1+\pi)\sin(\theta_1+\pi) - \cos\theta_1\sin\theta_1\right\}$$

$$\alpha_{11}(\xi) = \frac{-1}{4\pi(1-\nu)} \left\{(2-2\nu)\pi + \sin\theta_1\cos\theta_1 - \cos\theta_1\sin\theta_1\right\}$$

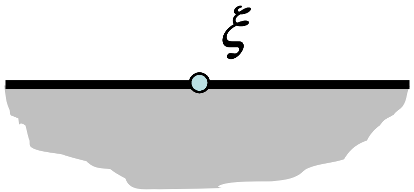
$$\alpha_{11}(\xi) = -\frac{1}{2}$$

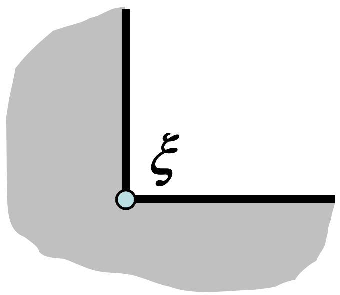
So $$c_{11} = \delta_{11} + \alpha_{11} = 0.5$$

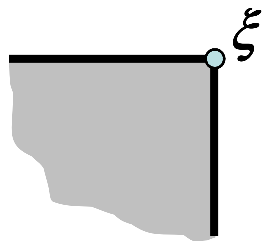
Boundary integral equation

Generally the 'free term' or 'jump term' $c_{ik}(\xi)$ is determined by the angle β subtended by the material at ξ.

$$c_{ik} = \frac{\beta}{2\pi}\delta_{ik}$$







$c_{ik} = 0.5\ \delta_{ik}$ $\qquad$ $c_{ik} = 0.75\ \delta_{ik}$ $\qquad$ $c_{ik} = 0.25\ \delta_{ik}$

Still need to compute $\lim_{\rho \to 0} \left\{ \int_{\Gamma - \Gamma_\varepsilon} T_{ik} u_i d\Gamma \right\}$

But…. much easier to use the row-sum method

Boundary integral equation

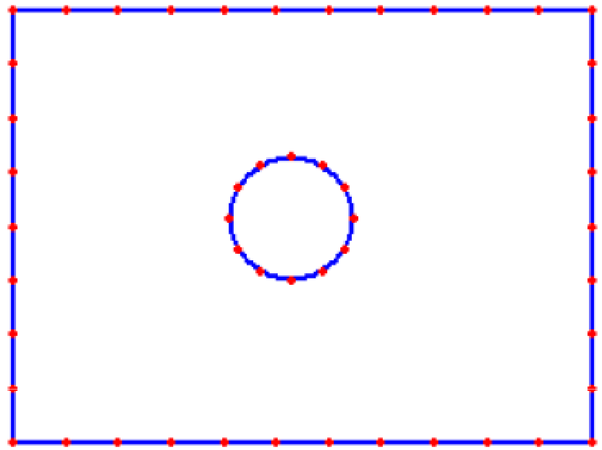
Boundary integral equation

$$c_{ik}(\xi)\, u_k(\xi) + \int_\Gamma T_{ik} u_i d\Gamma = \int_\Gamma U_{ik} t_i d\Gamma$$

Analytical solution is possible only for the very simplest problems. We will proceed by discretisation, leading to the ***Boundary Element Method*** itself.

Boundary element method

We discretise the boundary into elements.

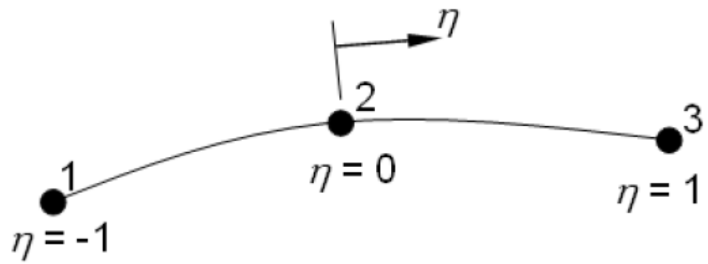


This provides:

- Nodal points and local interpolation to define an approximate solution, exactly like finite elements
- Convenient small portions of the boundary to perform numerical integration accurately
- A set of node points on which to place ξ in turn to provide a square system of linear equations

Boundary element method

Boundary elements share many essential characteristics with finite elements:



$$N_1(\eta) = \frac{\eta}{2}(\eta - 1)$$

$$N_2(\eta) = (1 - \eta)(1 + \eta)$$

$$N_3(\eta) = \frac{\eta}{2}(\eta + 1)$$

$$u_i(\eta) = \sum_{j=1}^{3} N_j(\eta)\, u_i^{jm}$$

Boundary element method

Discrete form of the boundary integral equation:

$$c_{ik}(\xi)\,u_k(\xi) + \sum_{m=1}^{M}\int_{\Gamma_m} T_{ik}u_i\,d\Gamma = \sum_{m=1}^{M}\int_{\Gamma_m} U_{ik}t_i\,d\Gamma$$

Express u and t in their interpolated forms over element m

$$c_{ik}(\xi)\,u_k(\xi) + \sum_{m=1}^{M}\int_{\Gamma_m} T_{ik}N_p u_i^{pm}\,d\Gamma = \sum_{m=1}^{M}\int_{\Gamma_m} U_{ik}N_p t_i^{pm}\,d\Gamma$$

Remove vectors of ***nodal*** displacements and tractions from the integrals:

$$c_{ik}(\xi)\,u_k(\xi) + \sum_{m=1}^{M}\int_{\Gamma_m} T_{ik}N_p\,d\Gamma\;u_i^{pm} = \sum_{m=1}^{M}\int_{\Gamma_m} U_{ik}N_p\,d\Gamma\;t_i^{pm}$$

Boundary element method

$$c_{ik}(\xi)\,u_k(\xi)+\sum_{m=1}^{M}\int_{\Gamma_m}T_{ik}N_p d\Gamma\;\,u_i^{pm}=\sum_{m=1}^{M}\int_{\Gamma_m}U_{ik}N_p d\Gamma\;\,t_i^{pm}$$

Transform boundary integrals to local coordinate system:

$$c_{ik}(\xi)\,u_k(\xi)+\sum_{m=1}^{M}\int_{-1}^{1}T_{ik}N_pJ(\eta)\,d\eta\;\,u_i^{pm}=\sum_{m=1}^{M}\int_{-1}^{1}U_{ik}N_pJ(\eta)\,d\eta\;\,t_i^{pm}$$

Evaluate this equation for ξ at, say, node 1 and Dirac force in x-direction:

$$c_{11}(1)\,u_1^1+\hat{h}_{1,1}u_1^1+\hat{h}_{1,2}u_2^1+\hat{h}_{1,3}u_1^2+\hat{h}_{1,4}u_2^2+\hat{h}_{1,5}u_1^3+\ldots$$
$$=g_{1,1}t_1^1+g_{1,2}t_2^1+g_{1,3}t_1^2+g_{1,4}t_2^2+g_{1,5}t_1^3+\ldots$$

Boundary element method

$$c_{11}(1)\,u_1^1 + \hat{h}_{1,1}u_1^1 + \hat{h}_{1,2}u_2^1 + \hat{h}_{1,3}u_1^2 + \hat{h}_{1,4}u_2^2 + \hat{h}_{1,5}u_1^3 + \ldots$$
$$= g_{1,1}t_1^1 + g_{1,2}t_2^1 + g_{1,3}t_1^2 + g_{1,4}t_2^2 + g_{1,5}t_1^3 + \ldots$$

Embed the free term c into the others….

$$h_{1,1}u_1^1 + h_{1,2}u_2^1 + h_{1,3}u_1^2 + h_{1,4}u_2^2 + h_{1,5}u_1^3 + \ldots$$
$$= g_{1,1}t_1^1 + g_{1,2}t_2^1 + g_{1,3}t_1^2 + g_{1,4}t_2^2 + g_{1,5}t_1^3 + \ldots$$

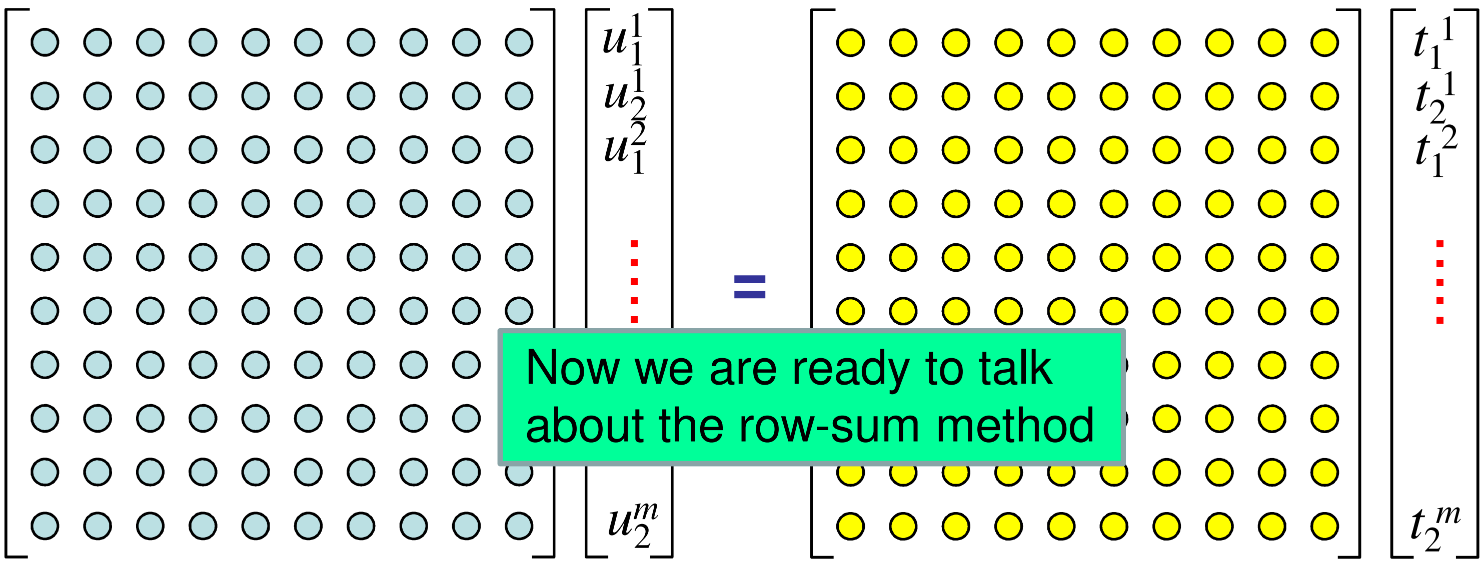
…by defining for notational simplicity

$$h_{i,j} = \hat{h}_{i,j} + \delta_{ij}c_{ij}(\xi)$$

Boundary element method

Evaluating the h and g terms for ξ at each node in turn gives

$$H\underline{u} = G\underline{t}$$



Matrix of h coefficients

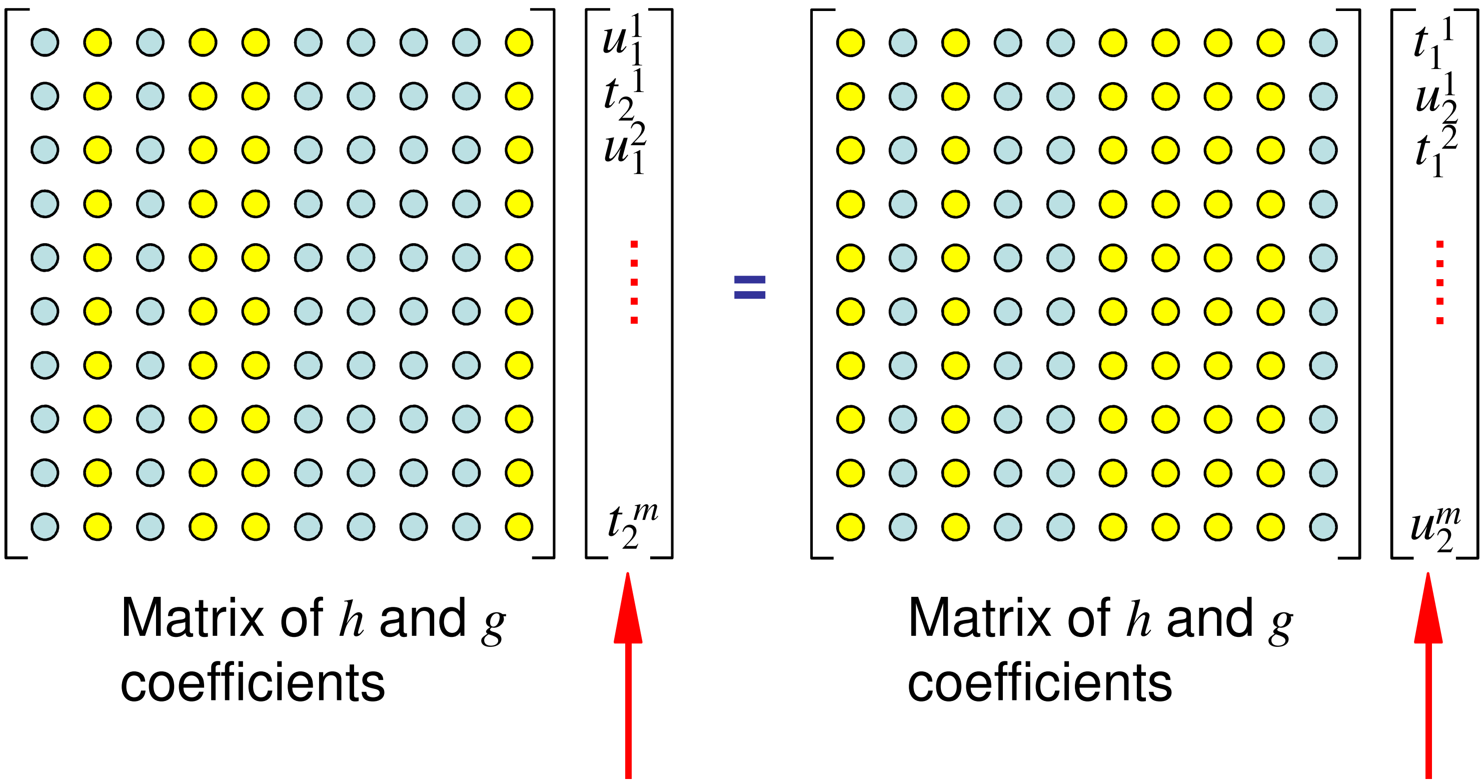
Matrix of g coefficients

Unknown displacements

Unknown tractions

Boundary element method

For each row, prescribe either the displacement or traction as a boundary condition, and column-swap to bring all remaining unknowns to LHS…



Matrix of h and g coefficients

Matrix of h and g coefficients

Unknown disp's & trac's

Boundary conditions

Boundary element method

The whole right hand side is now known and can be multiplied out leaving

$$A\underline{x} = \underline{b}$$

This can be solved either directly or iteratively.

Choice of solver is limited by asymmetry of A.

Boundary element method

Now move ξ off the boundary and into the material. We can solve for the displacements at this internal point using the equation $(*)$ we developed part way through the derivation

$$u_k(\xi) + \int_\Gamma T_{ik} u_i \, d\Gamma = \int_\Gamma U_{ik} t_i \, d\Gamma$$

Stress components at the internal point can be found from a derivative of this equation.

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Introduction

Reciprocal theorem

Fundamental solutions

Boundary Integral Equation

Boundary Element Method

Re-analysis and Interactivity

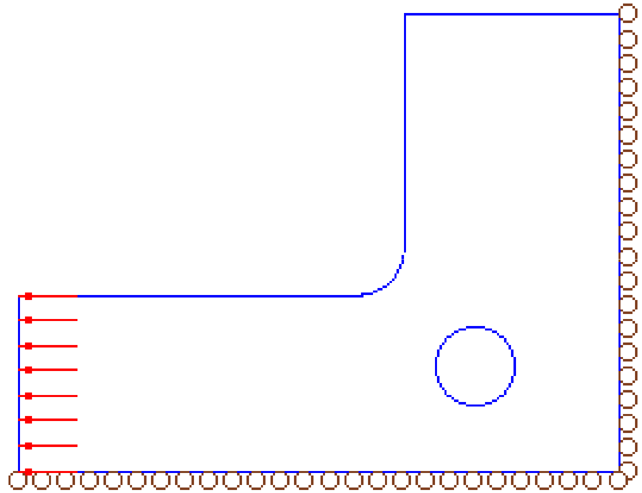
Demonstration

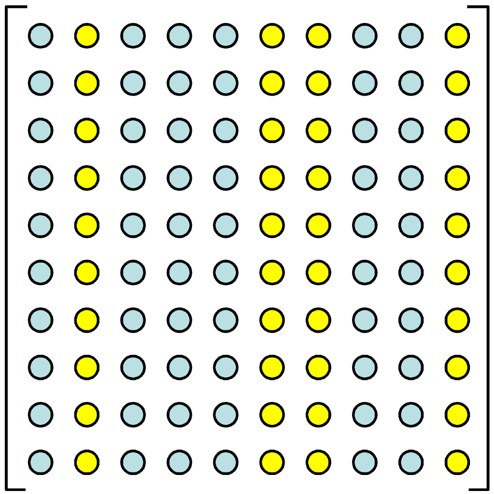
Enrichment

Re-analysis and interactivity

One aspect of the BEM we are pursuing in Durham is re-analysis leading to an interactivity to design analysis.

A typical problem in elasticity….





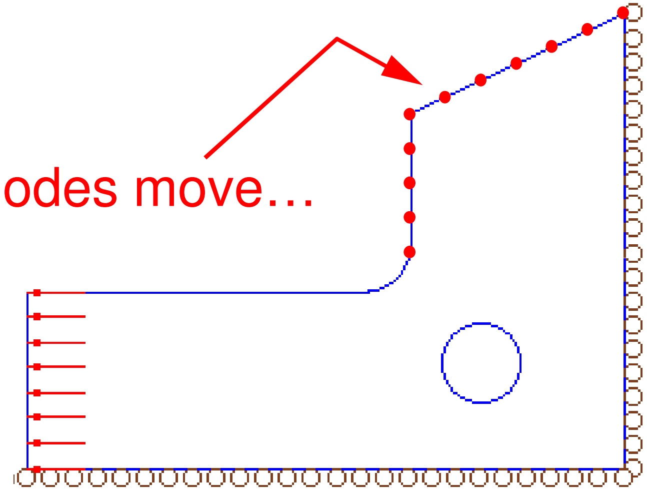
…and its matrix A

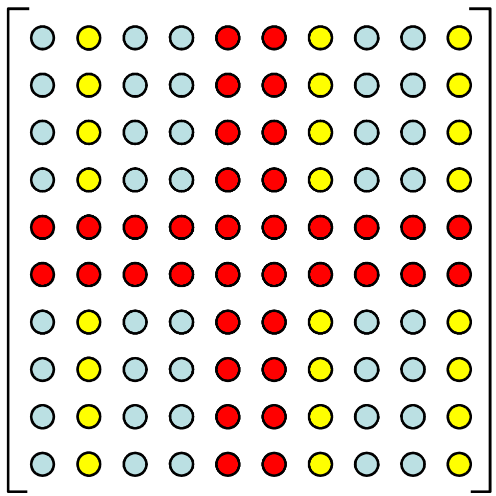
Re-analysis and interactivity

One aspect of the BEM we are pursuing here in Durham is re-analysis leading to an interactivity to design analysis.

Make a design change….

…only a few nodes move…

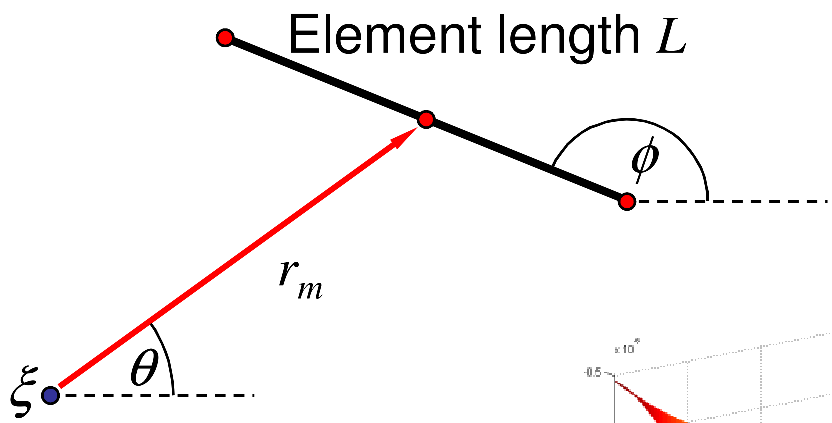


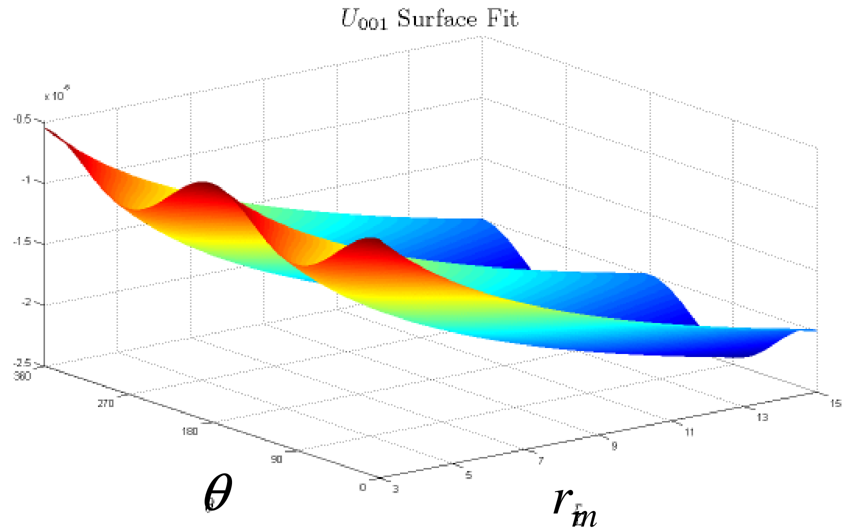


…and some rows and columns change, but most of the matrix is the same as the last one.

Surface fits for rapid approximation of integrals

$$U_{001} = \left[2.166\left(1+\cos\left(2\theta\right)\right) - 8.996\ln\left(\frac{r_m}{L}\right)\right] \times 10^{-7}$$





Iterative re-solution

We precondition a GMRES scheme with an approximate complete LU decomposition.

First analysis: LU-decomposition (save L and U)

Re-analysis: Use the full LU-decomposition of the original system as a preconditioner for iterative solution of the perturbed system.

Preconditioner	Perturbation Type				
	Move Point	Move circle	External Fillet Resize	Internal Fillet Resize 1	Internal Fillet Resize 2
None	30 – 50	31 – 49	36 – 48	34 – 50	37 – 53
Diagonal	39 – 53	36 – 47	36 – 44	43 – 47	44 – 50
Full LU	2 – 17	2 – 9	3 – 9	3 – 13	3 – 11

Number of iterations to convergence

Demonstration

Enriched approximation space

The shape functions form a Partition of Unity

$$\sum_{j=1}^{3} N_j(\xi) = 1$$

We use this property to enrich using arbitrary functions

$$\sum_{j=1}^{3} N_j(\xi)\psi(\xi) = \psi(\xi)$$

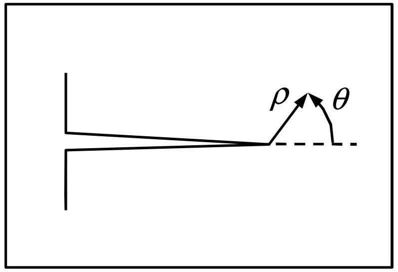
If we know functions ψ that populate the particular problem solution space we can include them in our approximation and obtain improved results .... Melenk & Babuška.

Enriched dual BEM for fracture mechanics

We can base an enrichment on the first order terms of the Williams expansion for displacement components around a crack tip

$$u_j^n(\xi) = \sum_{a=1}^{M} N_a(\xi) u_j^{na} + \sum_{a=1}^{M} \sum_{l=1}^{L} N_a(\xi) \psi_l^U(\xi) A_{jl}^{na}$$

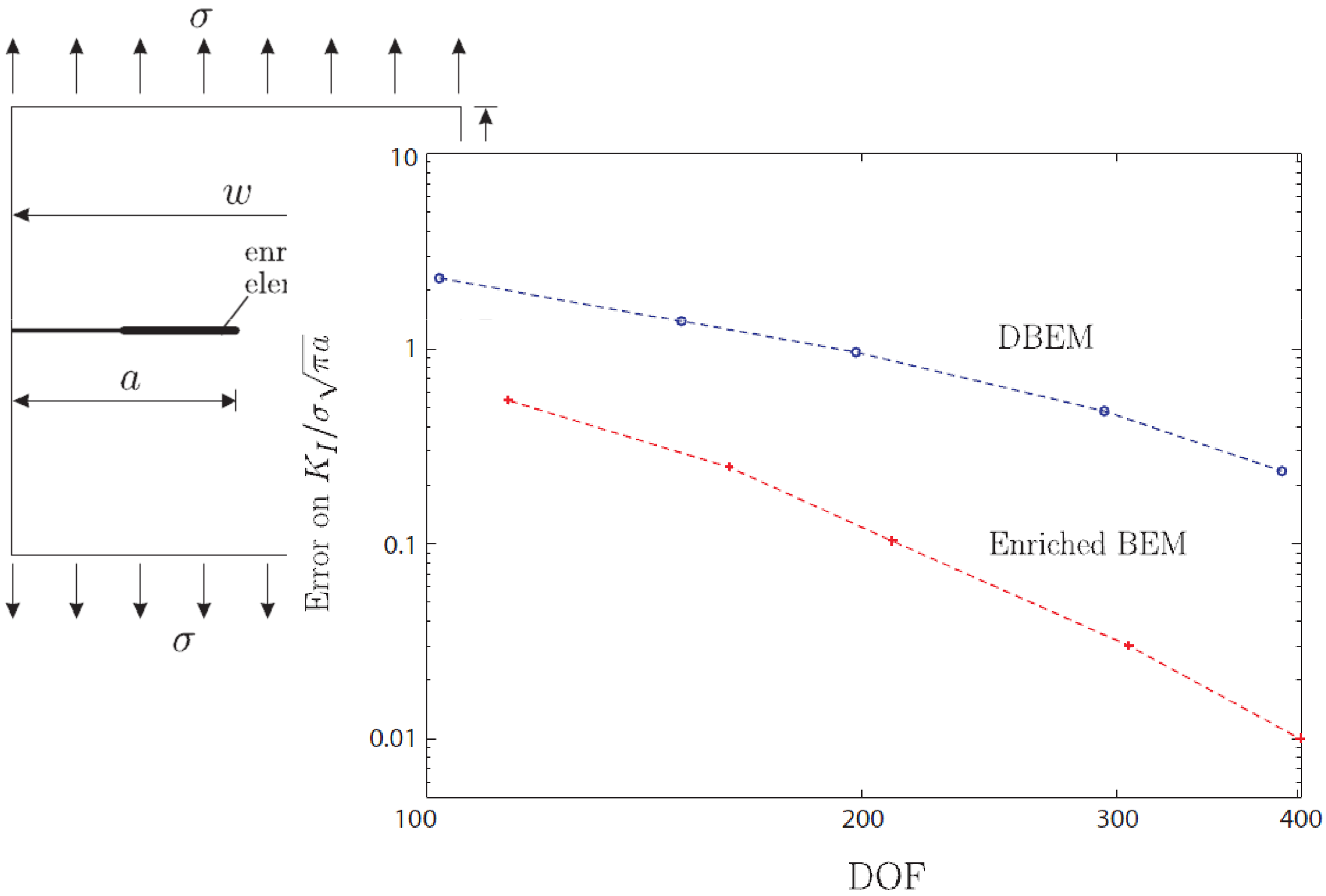
$$\psi^U(\rho,\theta) = \left\{ \sqrt{\rho}\cos\left(\frac{\theta}{2}\right), \sqrt{\rho}\sin\left(\frac{\theta}{2}\right), \sqrt{\rho}\sin\left(\frac{\theta}{2}\right)\sin(\theta), \sqrt{\rho}\cos\left(\frac{\theta}{2}\right)\sin(\theta) \right\}^{\mathbf{T}}$$



Remark: this is the same approximation space as used in the XFEM (Moës, Dolbow, Belytschko) but in a BEM sense.

Results – mode I problem

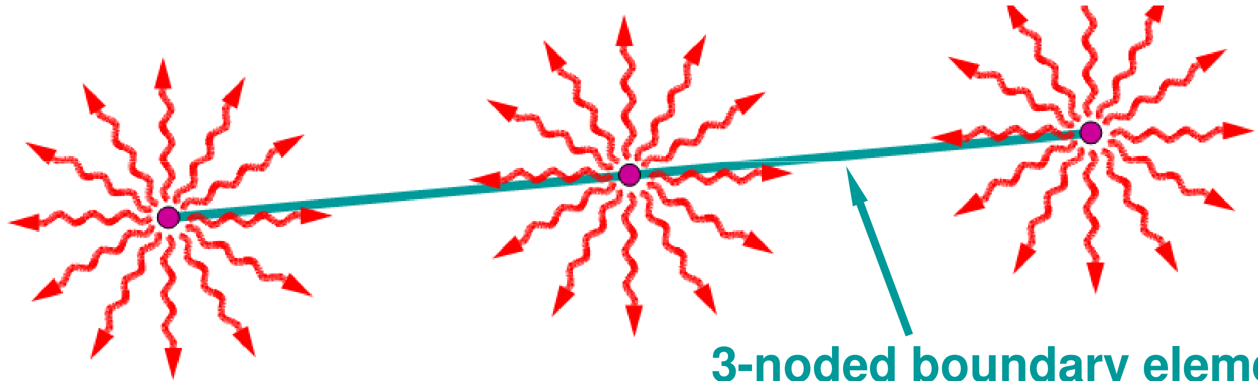
Edge crack in finite plate



PU-BEM enriched basis for wave problems

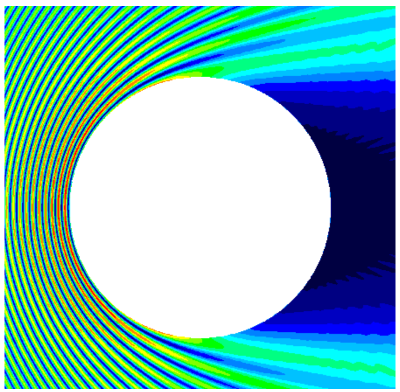
The PUM multiple plane wave expansion for potential on an element

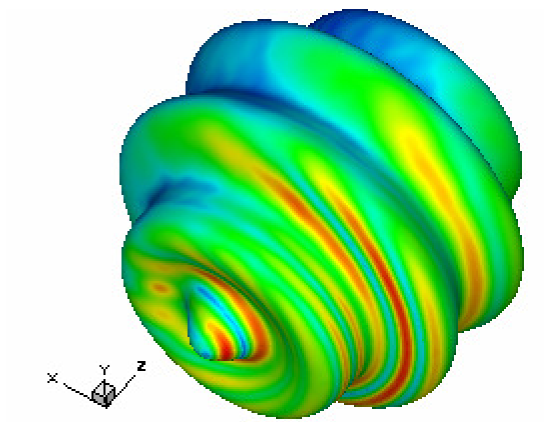
$$\phi(x) = \sum_{j=1}^{J} N_j(x) \sum_{m=1}^{M} A_{jm}^{e} e^{ik\psi_{jm}^{e}\cdot x}, \qquad |\psi_{jm}^{e}| = 1$$



12 waves at the node
$M = 12$

3-noded boundary element
$J = 3$





Conclusions

- BEM has been presented for elastostatics problems
- Body forces and non-linearity can be handled by further treatment not discussed here
- Attractive for various classes of problem
- Re-analysis leads to interactivity in stress analysis
- Enrichment of the approximation space can yield improved accuracy